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PERIODIC KALMAN FILTER: STEADY STATE FROM THE BEGINNING

MARIA ADAM[1] and NICHOLAS ASSIMAKIS[2]

[1]Department of Informatics with Applications to Biomedicine
University of Central Greece
35100 Lamia
Greece
e-mail: madam@ucg.gr

[2]Department of Informatics with Applications to Biomedicine
University of Central Greece
35100 Lamia
Greece

[2]Department of Electronics
Technological Educational Institute of Lamia
35100 Lamia
Greece

Abstract

A new approach for the periodic steady state Kalman filter is presented. The proposed algorithm requires the off-line solution of the corresponding periodic Riccati equation in order to take advantage of this a-priori knowledge (before the filter's implementation) using the steady state periodic gain from the beginning.

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1. Introduction

Estimation plays an important role in many fields of science. The discrete time Kalman filter [5] is the most well known algorithm that solves the estimation/filtering problem. Many real world problems have been successfully solved using the Kalman filter ideas; filter applications to aerospace industry, chemical process, communication systems design, control, civil engineering, filtering noise from 2-dimensional images, pollution prediction, power systems are mentioned in [1]. The study of steady state Kalman filter and its implementation for periodic models has attracted attention of many researchers and several results have been obtained [3], [8].

In this paper, it is presented a new approach for the steady state Kalman filter implementation for periodic models and proposed a new algorithm, which is faster than the classical implementation, due to the fact that the method requires only the knowledge of the steady state periodic gain from the beginning. The paper has been divided into the following sections: Section 2 presents the Steady state Kalman filter for periodic models; Section 3 presents the Proposed steady state Kalman filter for periodic models; Section 4 presents numerical examples based on simulations and compares the algorithms; Section 5 presents the conclusions.

2. Steady State Kalman Filter for Periodic Models

The estimation problem arises in linear estimation and is associated with time varying systems described by the following state space equations:

$$x(k+1) = F(k+1,\, k)x(k) + w(k), \tag{1}$$

$$z(k) = H(k)x(k) + v(k) \tag{2}$$

for $k = 0, 1, \ldots,$ where $x(k)$ is the n-dimensional state vector, $z(k)$ is the m-dimensional measurement vector, $F(k+1,\, k)$ is the $n \times n$ system transition matrix, $H(k)$ is the $m \times n$ output matrix, $\{w(k)\}$ and $\{v(k)\}$ are the plant noise and the measurement noise processes; they are assumed to be Gaussian, zero-mean white and uncorrelated random processes and $Q(k)$, $R(k)$ are the plant noise and measurement noise covariance matrices, respectively. The vector $x(0)$ is a Gaussian random process with mean x_0 and covariance P_0 and $x(0)$, $\{w(k)\}$ and $\{v(k)\}$ are independent.

The filtering/estimation problem is to produce an estimate at time L of the state vector using measurements till time L, i.e., the aim is to use the measurements set $\{z(1), \ldots, z(L)\}$ in order to calculate an estimate value $x(L/L)$ of the state vector $x(L)$. The discrete time Kalman filter is summarized in the following:

Kalman Filter (KF)**.**

$$K(k) = P(k/k-1)H^T(k)[H(k)P(k/k-1)H^T(k) + R(k)]^{-1}, \tag{3}$$

$$x(k/k) = [I - K(k)H(k)]x(k/k-1) + K(k)z(k), \tag{4}$$

$$P(k/k) = [I - K(k)H(k)]P(k/k-1), \tag{5}$$

$$x(k+1/k) = F(k+1, k)x(k/k), \tag{6}$$

$$P(k+1/k) = Q(k) + F(k+1, k)P(k/k)F^T(k+1, k) \tag{7}$$

for $k = 0, 1, \ldots,$ with initial conditions $x(0/-1) = x_0$ and $P(0/-1) = P_0$, where $x(k/k)$ is the estimate value of the state vector, $P(k/k)$ is the corresponding estimation error covariance matrix, $K(k)$ is the Kalman filter gain, $x(k+1/k)$ is the prediction value of the state vector, $P(k+1/k)$ is the corresponding prediction error covariance matrix.

In the case of periodic model, the matrices $F(k+1, k)$, $H(k)$, $Q(k)$ and $R(k)$ are periodic with period p, i.e.,

$$F(k+1+ip, k+ip) = F(k+1+(i+1)p, k+(i+1)p), \tag{8}$$

$$H(k+ip) = H(k+(i+1)p), \tag{9}$$

$$Q(k+ip) = Q(k+(i+1)p), \tag{10}$$

$$R(k + ip) = R(k + (i + 1)p) \tag{11}$$

for $k = 0, 1, \ldots, p - 1,$ and $i = 0, 1, \ldots$.

The corresponding discrete time periodic Riccati equation [8] resulting from equations (3), (5) and (7) is as follows:

$$P(k+1/k) = Q(k) + F(k+1, k)P(k/k-1)F^T(k+1, k)$$
$$- F(k+1, k)P(k/k-1)H^T(k)[H(k)P(k/k-1)H^T(k)$$
$$+ R(k)]^{-1}H(k)P(k/k-1)F^T(k+1, k). \tag{12}$$

It is known [8] for periodic systems that the discrete time periodic Riccati equation has a steady state periodic stabilizing solution with period p :

$$\overline{P}(k + 1 + ip / k + ip) = \overline{P}(k + 1 + (i + 1)p/k + (i + 1)p). \tag{13}$$

Then, it is obvious that in the Kalman filter the gain matrix becomes periodic with period p :

$$\overline{K}(k + 1 + ip) = \overline{K}(k + 1 + (i + 1)p), \tag{14}$$

where

$$\overline{K}(k+1) = \overline{P}(k+1/k)H^T(k+1)[H(k+1)\overline{P}(k+1/k)H^T(k+1)$$
$$+ R(k+1)]^{-1}. \tag{15}$$

Combining equations (4) and (6) we are able to write:

$$x(k+1/k+1) = A(k+1, k)x(k/k) + K(k+1)z(k+1), \tag{16}$$

where

$$A(k+1, k) = [I - K(k+1)H(k+1)]F(k+1, k). \tag{17}$$

We observe that the matrix $A(k+1, k)$ becomes periodic with period p :

$$\overline{A}(k + 1 + ip, k + ip) = \overline{A}(k + 1 + (i + 1)p, k + (i + 1)p), \tag{18}$$

where

$$\overline{A}(k+1, k) = [I - \overline{K}(k+1)H(k+1)]F(k+1, k). \tag{19}$$

Thus, in the steady state case, after the steady state time is reached in s periods, the resulting discrete time steady state periodic Kalman filter is as follows:

Steady State Periodic Kalman Filter (SSPKF)**.**

$$x(sp + k + 1/sp + k + 1) = \overline{A}(k \bmod p + 1,\ k \bmod p)x(sp + k/sp + k) + \overline{K}(k \bmod p + 1)z(sp + k + 1) \tag{20}$$

for $k = 0, 1, \ldots$.

Remark 1. The Steady State Periodic Kalman Filter (SSPKF) implementation requires the Kalman filter implementation for $k = 0, 1, \ldots, sp$ in order to calculate the estimate $x(sp/sp)$.

Remark 2. The steady state periodic prediction error covariance $\overline{P}(k + 1, k)$ is calculated by off-line solving the corresponding discrete time periodic Riccati equation (12) using techniques for solving the discrete time Riccati equation [1, 4, 6, 7, 8]. In the following, the steady state periodic gain matrix $\overline{K}(k + 1)$ and the corresponding matrix $\overline{A}(k + 1, k)$ are calculated off-line using (15) and (19), respectively.

3. Proposed Steady State Kalman Filter for Periodic Models

We are now ready to present a new approach for the steady state periodic Kalman filter. The method is based on the proposed algorithm in [2] concerning non-periodic models. It requires the off-line solution of the corresponding periodic Riccati equation (12). The proposed Alpha Steady State Periodic Kalman Filter algorithm is summarized in the following:

Alpha Steady State Periodic Kalman Filter (ASSPKF)**.**

$$x(k + 1/k + 1) = \overline{A}(k \bmod p + 1,\ k \bmod p)x(k/k) + \overline{K}(k \bmod p + 1)z(k + 1) \tag{21}$$

for $k = 0, 1, \ldots,$ with initial condition $x(0/0) = [I - K(0)H(0)]x_0 + K(0)z(0)$, where $K(0) = P_0H^T(0)[H(0)P_0H^T(0) + R(0)]^{-1}$.

Remark 3. This steady state Kalman filter (ASSPKF) implementation **does not require** the Kalman filter implementation.

Remark 4. As Remark 2, the steady state periodic prediction error covariance $\overline{P}(k+1, k)$ is calculated by off-line solving the corresponding discrete time periodic Riccati equation (12) using techniques for solving the discrete time Riccati equation [1, 4, 6, 7, 8]. In the following, the steady state periodic gain matrix $\overline{K}(k+1)$ and the corresponding matrix $\overline{A}(k+1, k)$ are calculated off-line using (15) and (19), respectively.

Remark 5. The classical algorithm (SSPKF) requires the implementation of equations (3)-(7) (Kalman filter) for $k = 0, 1, \ldots, sp$ in order to calculate the estimate $x(sp/sp)$ and the implementation of equation (20) for $k = 0, 1, \ldots,$ while the proposed algorithm (ASSPKF) requires the implementation of equation (21) for $k = 0, 1, \ldots$ (from the beginning). It is obvious that the computation times needed for the implementation (on-line calculations) of both equations (20) and (21) are equivalent to each other. It is also obvious that the computation time needed for the implementation of equation (21) is less than the computation time needed for the implementation of equations (3)-(7). Thus, the proposed algorithm is faster than the classical one.

Remark 6. The method is based on taking advantage of the a-priori knowledge of the steady state periodic gain and using it in the Kalman filter equations from the beginning $(k = 0)$. This is the reason we named the proposed algorithm "Alpha Steady State Periodic Kalman Filter", in order to show the use of the steady state gain from the beginning (alpha is the first letter of the Greek alphabet).

4. Simulation Results

Simulation experiments were carried out in order to investigate the behavior of the proposed algorithm in comparison to the classical algorithm's behavior. Various values for the state vector dimension n, the measurement vector dimension m and the period p were considered. It was verified that the proposed algorithm presents a very good behavior compared to the classical one.

Example 7. In the following example we consider the case $n = 1$ and $m = 1$ (a simple scalar case) with period $p = 2$, where $F(1, 0) = 0.6$ and $F(2, 1) = 0.9$, $H(0) = 1.2$ and $H(1) = 1.4$, $Q(0) = 0.4$ and $Q(1) = 0.1$, $R(0) = 0.3$ and $R(1) = 0.2$, with initial conditions $x(0/-1) = x_0 = 0$ and $P(0/-1) = P_0 = 0$. The steady state periodic prediction error covariance is:

$$\overline{P}(1/0) = 0.1669,\ \overline{P}(2/1) = 0.4334,$$

and the steady state time is reached in $s = 7$ periods. The steady state periodic gain is:

$$\overline{K}(1) = 0.3707,\ \overline{K}(2) = 0.5782$$

and

$$\overline{A}(1, 0) = 0.4997,\ \overline{A}(2, 1) = 0.1144.$$

In order to compare the algorithms we compute the estimate values $x(k/k)$ of the state vector $x(k)$ for $k = 0, 1, ..., 100$ using both the classical and the proposed algorithms. The state $x(k)$ as well as the calculated estimates using SSPKF and ASSPKF are plotted in Figure 1. The estimates using SSPKF and ASSPKF are very close to each other: the maximum percent absolute error is $3.7719 \cdot 10^{-4}\%$.

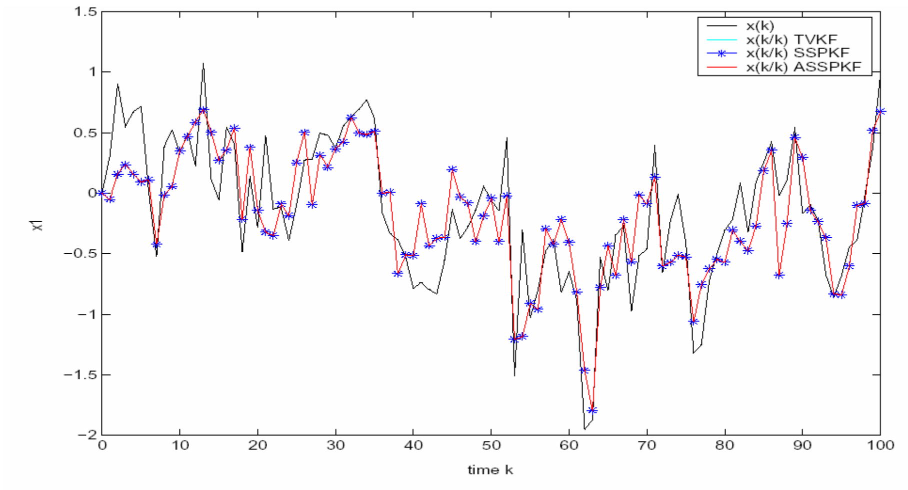

Figure 1. State $x(k)$ and estimates $x(k/k)$ using SSPKF and ASSPKF for Example 7.

Example 8. In the following example we consider the case $n = 1$ and $m = 2$ with period $p = 2$, where $F(1, 0) = 0.6$ and $F(2, 1) = 0.9$, $H(0) = \begin{bmatrix} 1.2 \\ 1.8 \end{bmatrix}$ and $H(1) = \begin{bmatrix} 1.4 \\ 1.6 \end{bmatrix}$, $Q(0) = 0.4$ and $Q(1) = 0.1$, $R(0) = \begin{bmatrix} 0.3 & 0 \\ 0 & 1 \end{bmatrix}$ and $R(1) = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}$, with initial conditions $x(0/-1) = x_0 = 0$ and $P(0/-1) = P_0 = 0$. The steady state periodic prediction error covariance is:

$$\overline{P}(1/0) = 0.1551, \ \overline{P}(2/1) = 0.4248,$$

and $s = 7$. The state $x(k)$ as well as the calculated estimates using SSPKF and ASSPKF are plotted in Figure 2. The estimates using SSPKF and ASSPKF are very close to each other.

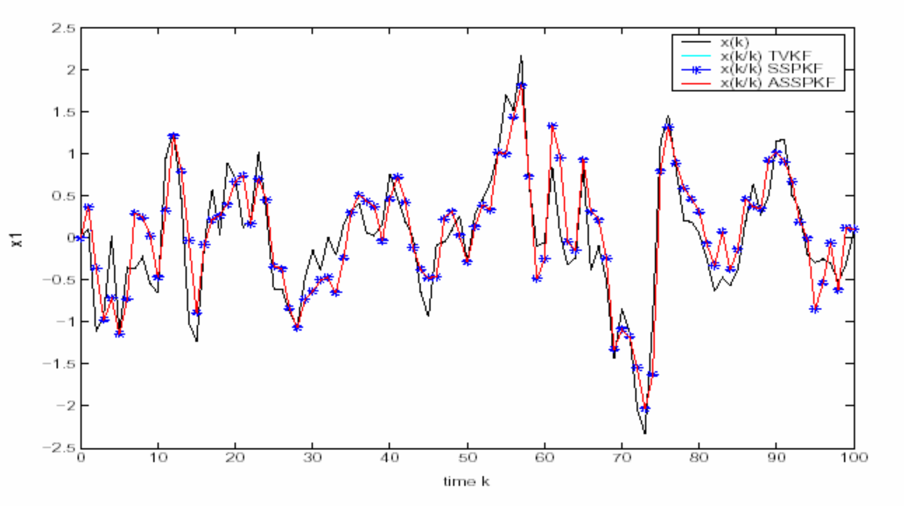

Figure 2. State $x(k)$ and estimates $x(k/k)$ using SSPKF and ASSPKF for Example 8.

Example 9. We consider the case $n = 2$ and $m = 1$ with period $p = 2$, where $F(1, 0) = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.9 \end{bmatrix}$ and $F(2, 1) = \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$, $H(0) = [1.2 \quad 1.4]$ and $H(1) = [1.1 \quad 1.5]$, $Q(0) = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.1 \end{bmatrix}$ and $Q(1) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix}$, $R(0) = 0.3$ and $R(1) = 0.2$, with initial conditions $x(0/-1) = x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $P(0/-1) = P_0 = \mathbb{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The steady state periodic prediction error covariance is:

$$\overline{P}(1/0) = \begin{bmatrix} 0.5492 & -0.0156 \\ -0.0156 & 0.2465 \end{bmatrix}, \overline{P}(2/1) = \begin{bmatrix} 0.4722 & 0.0051 \\ 0.0051 & 0.1785 \end{bmatrix},$$

and $s = 12$. The state $x(k)$ as well as the calculated estimates using SSPKF and ASSPKF are plotted in Figure 3.

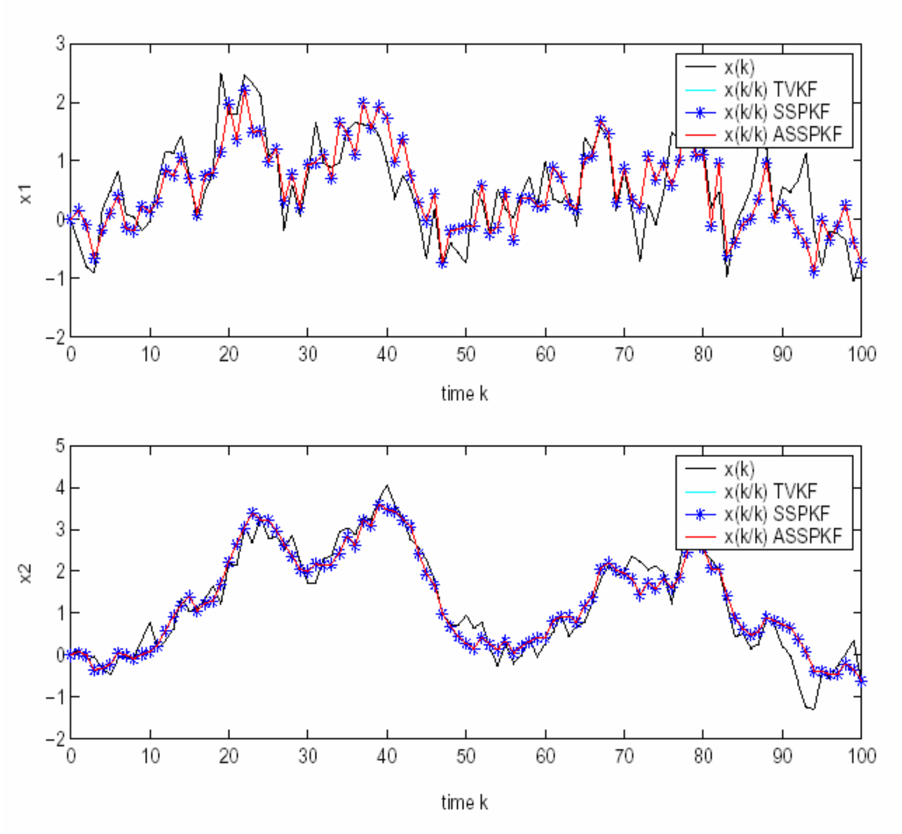

Figure 3. State $x(k)$ and estimates $x(k/k)$ using SSPKF and ASSPKF for Example 9.

Example 10. We consider the case $n = 2$ and $m = 2$ with period $p = 2$, where $F(1, 0) = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.9 \end{bmatrix}$ and $F(2, 1) = \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$, $H(0) = \begin{bmatrix} 1.2 & 1.4 \\ 1.8 & 1.6 \end{bmatrix}$ and $H(1) = \begin{bmatrix} 1.1 & 1.5 \\ 1.2 & 1.3 \end{bmatrix}$, $Q(0) = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.1 \end{bmatrix}$ and $Q(1) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix}$, $R(0) = \begin{bmatrix} 0.3 & 0 \\ 0 & 1 \end{bmatrix}$ and $R(1) = \begin{bmatrix} 0.2 & 0 \\ 0 & 2 \end{bmatrix}$, with initial conditions $x(0/-1) = x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $P(0/-1) = P_0 = \mathbb{O}$.

The steady state periodic prediction error covariance is:

$$\overline{P}(1/0) = \begin{bmatrix} 0.5470 & -0.0168 \\ -0.0168 & 0.2453 \end{bmatrix}, \ \overline{P}(2/1) = \begin{bmatrix} 0.4548 & -0.0071 \\ -0.0071 & 0.1697 \end{bmatrix},$$

and $s = 13$. The state $x(k)$ as well as the calculated estimates using SSPKF and ASSPKF are plotted in Figure 4.

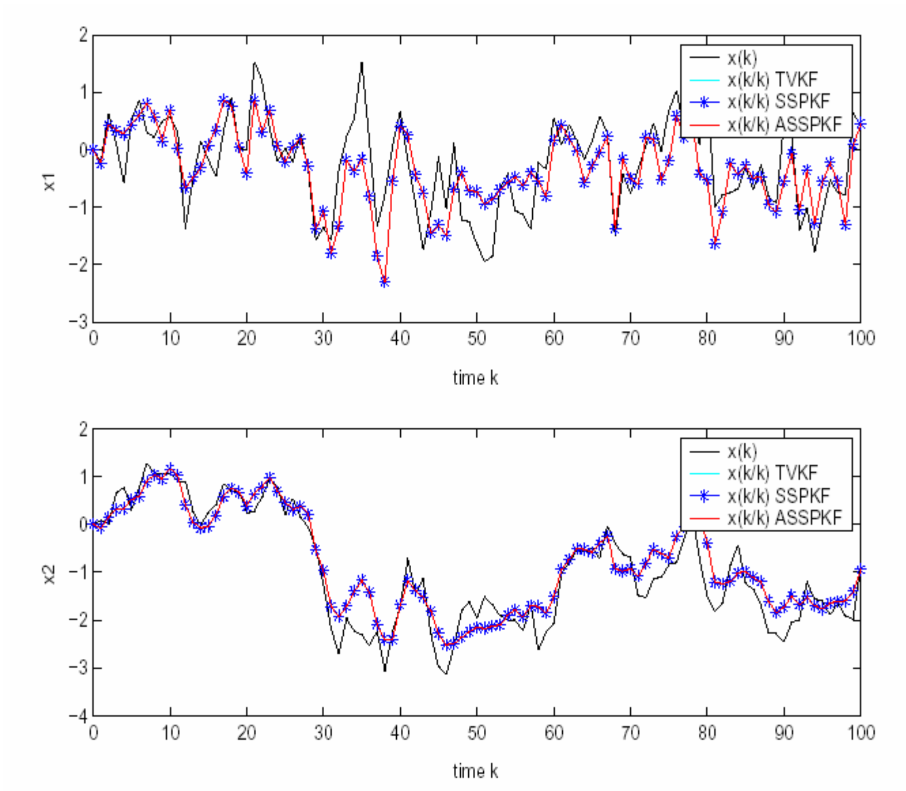

Figure 4. State $x(k)$ and estimates $x(k/k)$ using SSPKF and ASSPKF for Example 10.

Example 11. In the following example we consider a simple scalar case ($n = 1$ and $m = 1$) with period $p = 3$, where $F(1, 0) = 0.6$, $F(2, 1) = 0.9$ and $F(3, 2) = 0.8$, $H(0) = 1.2$, $H(1) = 1.4$ and $H(2) = 1.1$, $Q(0) = 0.4$, $Q(1) = 0.1$ and $Q(2) = 0.2$, $R(0) = 0.3$, $R(1) = 0.2$ and $R(2) = 0.4$, with zero initial conditions $x(0/-1) = x_0 = 0$ and $P(0/-1) = P_0 = 0$. The steady state periodic prediction error covariance is:

$$\overline{P}(1/0) = 0.1672,\ \overline{P}(2/1) = 0.2711,\ \overline{P}(3/2) = 0.4424,$$

and $s = 6$. The state $x(k)$ as well as the calculated estimates using SSPKF and ASSPKF are plotted in Figure 5.

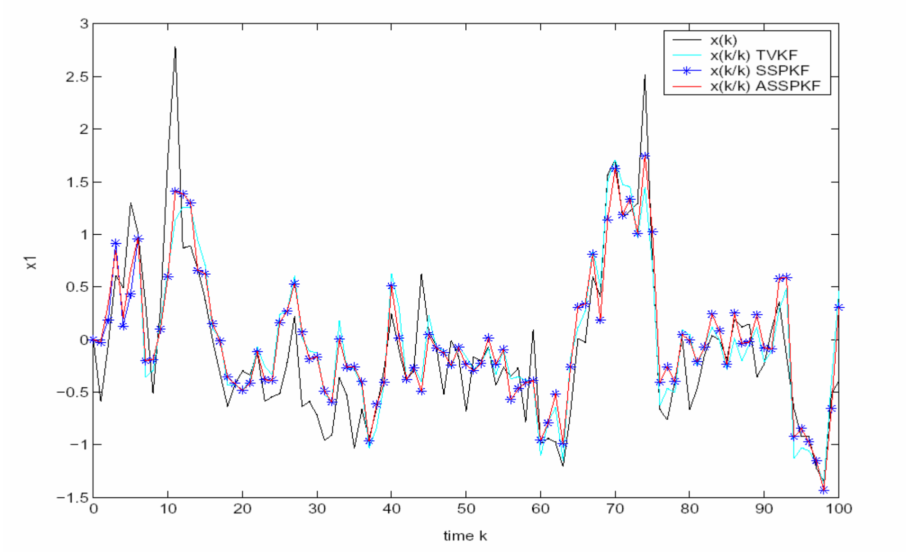

Figure 5. State $x(k)$ and estimates $x(k/k)$ using SSPKF and ASSPKF for Example 11.

Example 12. In the following example we consider the same data as in Example 11, i.e., $n = 1$ and $m = 1$, with period $p = 3$, where $F(1, 0) = 0.6$, $F(2, 1) = 0.9$ and $F(3, 2) = 0.8$, $H(0) = 1.2$, $H(1) = 1.4$ and $H(2) = 1.1$, $Q(0) = 0.4$, $Q(1) = 0.1$ and $Q(2) = 0.2$, $R(0) = 0.3$, $R(1) = 0.2$ and $R(2) = 0.4$, with nonzero initial conditions $x(0/-1) = x_0 = 2$ and $P(0/-1) = P_0 = 1$. The steady state periodic prediction error covariance is the same as in Example 11.

$$\overline{P}(1/0) = 0.1672,\ \overline{P}(2/1) = 0.2711,\ \overline{P}(3/2) = 0.4424,$$

and $s = 6$. The state $x(k)$ as well as the calculated estimates using SSPKF and ASSPKF are plotted in Figure 6.

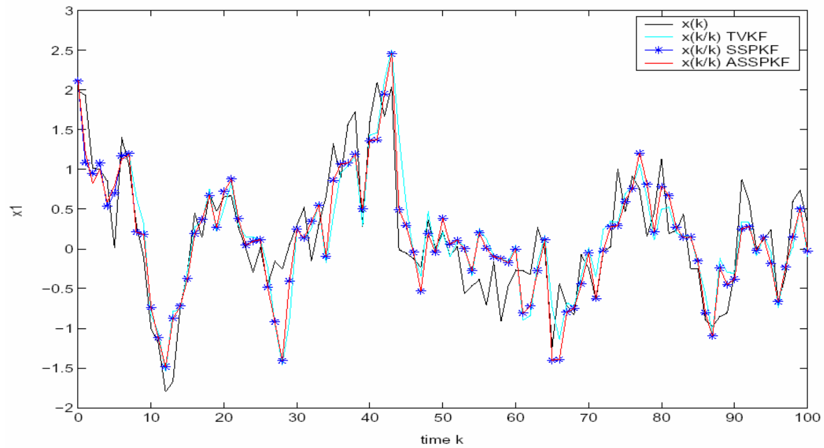

Figure 6. State $x(k)$ and estimates $x(k/k)$ using SSPKF and ASSPKF for Example 12.

Example 13. We consider the case $n = 1$ and $m = 2$, with period $p = 3$, where $F(1, 0) = F(2, 1) = F(3, 2) = 0.6$, $H(0) = \begin{bmatrix} 1.2 \\ 1.8 \end{bmatrix}$, $H(1) = \begin{bmatrix} 1.4 \\ 1.6 \end{bmatrix}$ and $H(2) = \begin{bmatrix} 1.5 \\ 1.9 \end{bmatrix}$, $Q(0) = Q(1) = Q(2) = 0.4$, $R(0) = \begin{bmatrix} 0.3 & 0 \\ 0 & 1 \end{bmatrix}$, $R(1) = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}$ and $R(2) = \begin{bmatrix} 0.4 & 0 \\ 0 & 1 \end{bmatrix}$, with initial conditions $x(0/-1) = x_0 = 1$ and $P(0/-1) = P_0 = 1$. The steady state periodic prediction error covariance is:

$$\overline{P}(1/0) = 0.4246,\ \overline{P}(2/1) = 0.4311,\ \overline{P}(3/2) = 0.4347,$$

and $s = 4$. The state $x(k)$ as well as the calculated estimates using SSPKF and ASSPKF are plotted in Figure 7.

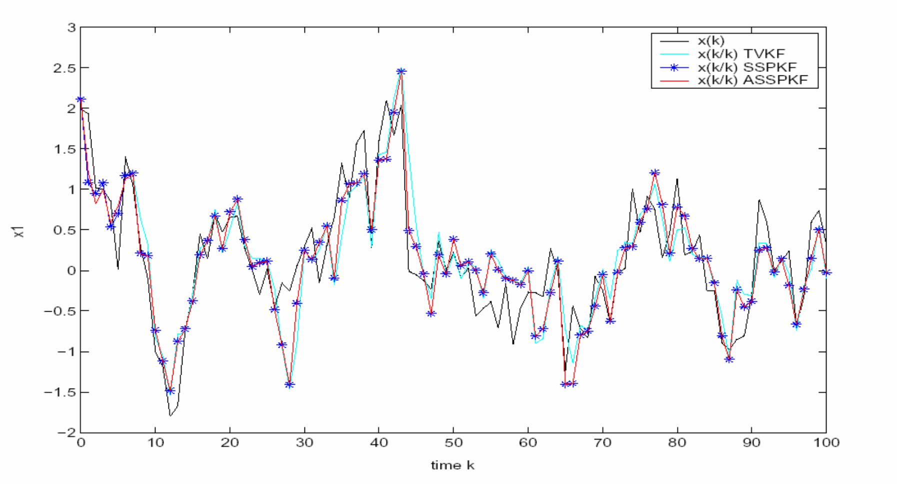

Figure 7. State $x(k)$ and estimates $x(k/k)$ using SSPKF and ASSPKF for Example 13.

Example 14. We consider the case $n = 2$ and $m = 1$, with period $p = 3$, where $F(1, 0) = F(2, 1) = F(3, 2) = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$, $H(0) = H(1) = H(2) = [1.2 \quad 1.4]$, $Q(0) = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.1 \end{bmatrix}$, $Q(1) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix}$ and $Q(2) = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.4 \end{bmatrix}$, $R(0) = 0.3$, $R(1) = 0.2$ and $R(2) = 0.5$ with initial conditions $x(0/-1) = x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $P(0/-1) = P_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The steady state periodic prediction error covariance is:

$$\overline{P}(1/0) = \begin{bmatrix} 0.5522 & 0.0072 \\ 0.0072 & 0.2416 \end{bmatrix}, \overline{P}(2/1) = \begin{bmatrix} 0.3854 & 0.0310 \\ 0.0310 & 0.4809 \end{bmatrix},$$

$$\overline{P}(3/2) = \begin{bmatrix} 0.4610 & 0.0062 \\ 0.0062 & 0.1769 \end{bmatrix}$$

and $s = 7$. The state $x(k)$ as well as the calculated estimates using SSPKF and ASSPKF are plotted in Figure 8.

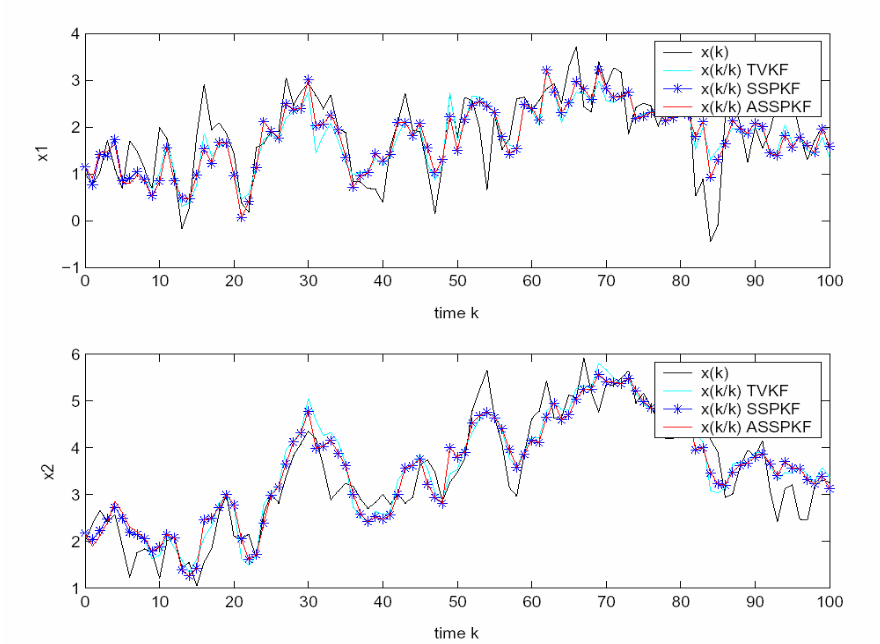

Figure 8. State $x(k)$ and estimates $x(k/k)$ using SSPKF and ASSPKF for Example 14.

Table 1 summarizes the maximum percent absolute error for all the examples: the proposed algorithm presents a very good behavior compared to the classical one.

Table 1. Maximum percent absolute error

Example	State vector dimension n	Measurement vector dimension m	Period p	Initial conditions	Maximum percent absolute error (%)
7	1	1	2	zero	$3.7719 \cdot 10^{-4}$
8	1	2	2	zero	$2.4000 \cdot 10^{-3}$
9	2	1	2	zero	$1.4000 \cdot 10^{-3}$
10	2	2	2	zero	$5.2000 \cdot 10^{-3}$
11	1	1	3	zero	$1.9220 \cdot 10^{-1}$
12	1	1	3	nonzero	$1.5500 \cdot 10^{-2}$
13	1	2	3	nonzero	$2.5829 \cdot 10^{-4}$
14	2	1	3	nonzero	$3.8200 \cdot 10^{-2}$

5. Conclusions

A new algorithm for the steady state periodic Kalman filter is presented in this paper. The proposed method involves the off-line solution of the corresponding periodic Riccati equation and uses the a-priori (before the filter's implementation) calculated steady state periodic gain from the beginning. It was pointed out that the proposed implementation is faster than the classical one. It was finally verified that the proposed algorithm presents a very good behavior compared to the classical one.

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