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Steady State Kalman Filter for Periodic Models: A New Approach

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Abstract

In this paper a new approach for the steady state Kalman filter implementation for periodic models is proposed: the method requires the knowledge of a subset of previous time measurements in order to calculate the state estimate; there is no need of any previous estimates calculation. The proposed algorithm is fast and has a parallel structure.

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1 Steady state Kalman filter for periodic models

Estimation plays an important role in many fields of science. The discrete time Kalman filter [5] is the most well known algorithm that solves the estimation/filtering problem. Many real world problems have been successfully solved using the Kalman filter ideas; filter applications to aerospace industry, chemical process, communication systems design, control, civil engineering, filtering noise from 2−dimensional images, pollution prediction, power systems are mentioned in [1]. The estimation problem arises in linear estimation and is associated with time varying systems described by the following state space equations:

$$x(k+1) = F(k+1,k)x(k) + w(k) \tag{1}$$

$$z(k+1) = H(k+1)x(k+1) + v(k+1) \tag{2}$$

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where $x(k)$ is the $n-$dimensional state vector at time k, $z(k)$ is the $m-$dimensional measurement vector, $F(k+1,k)$ is the $n \times n$ system transition matrix, $H(k+1)$ is the $m \times n$ output matrix, $\{w(k)\}$ and $\{v(k)\}$ are Gaussian zero-mean white and uncorrelated random processes, $Q(k)$ and $R(k+1)$ are the plant noise and measurement noise covariance matrices, respectively. The vector $x(0)$ is a Gaussian random process with mean x_0 and covariance P_0 and $x(0)$, $\{w(k)\}$ and $\{v(k)\}$ are independent.

The filtering/estimation problem is to produce an estimate at time L of the state vector using measurements till time L, i.e. the aim is to use the measurements set $\{z(1), \ldots, z(L)\}$ in order to calculate an estimate value $x(L/L)$ of the state vector $x(L)$. The discrete time Kalman filter is summarized in the following:

Kalman Filter (KF)

$$K(k) = P(k/k-1)H^T(k)\left[H(k)P(k/k-1)H^T(k) + R(k)\right]^{-1} \tag{3}$$
$$P(k/k) = \left[I - K(k)H(k)\right]P(k/k-1) \tag{4}$$
$$P(k+1/k) = Q(k) + F(k+1,k)P(k/k)F^T(k+1,k) \tag{5}$$
$$x(k/k) = \left[I - K(k)H(k)\right]x(k/k-1) + K(k)z(k) \tag{6}$$
$$x(k+1/k) = F(k+1,k)x(k/k) \tag{7}$$

for $k = 0, 1, \ldots,$ with initial conditions $x(0/-1) = x_0$ and $P(0/-1) = P_0$. In the case of periodic model, the matrices $F(k+1,k)$, $H(k+1)$, $Q(k)$ and $R(k+1)$ are periodic with period p, i.e.:

$$F(k+1+ip, k+ip) = F(k+1+(i+1)p, k+(i+1)p) \tag{8}$$
$$H(k+1+ip) = H(k+1+(i+1)p) \tag{9}$$
$$Q(k+ip) = Q(k+(i+1)p) \tag{10}$$
$$R(k+1+ip) = R(k+1+(i+1)p) \tag{11}$$

for $k = 0, 1, \ldots, p-1$ and $i = 0, 1, \ldots$

The corresponding discrete time periodic Riccati equation [8] results from equations (3)-(5) and has as follows:

$$\begin{aligned} P(k+1/k) = {} & Q(k) + F(k+1,k)P(k/k-1)F^T(k+1,k) \\ & - F(k+1,k)P(k/k-1)H^T(k)\left[H(k)P(k/k-1)H^T(k) + R(k)\right]^{-1} \\ & H(k)P(k/k-1)F^T(k+1,k) \end{aligned} \tag{12}$$

It is known [8] for periodic systems that the discrete time periodic Riccati equation has a steady state periodic stabilizing solution with period p:

$$\bar{P}(k+1+ip/k+ip) = \bar{P}(k+1+(i+1)p/k+(i+1)p). \tag{13}$$

It is then obvious that the Kalman filter gain matrix $K(k)$ becomes periodic with period p:

$$\bar{K}(k+1+ip) = \bar{K}(k+1+(i+1)p), \tag{14}$$

where

$$\bar{K}(k) = \bar{P}(k/k-1)H^T(k)\left[H(k)\bar{P}(k/k-1)H^T(k)+R(k)\right]^{-1}. \tag{15}$$

Combining equations (6)-(7) we are able to write:

$$x(k+1/k+1) = A(k+1,k)\,x(k/k) + K(k+1)\,z(k+1), \tag{16}$$

where

$$A(k+1,k) = [I - K(k+1)H(k+1)]\,F(k+1,k). \tag{17}$$

We observe that the matrix $A(k+1,k)$ becomes periodic with period p:

$$\bar{A}(k+1+ip, k+ip) = \bar{A}(k+1+(i+1)p, k+(i+1)p), \tag{18}$$

where

$$\bar{A}(k+1,k) = F(k+1,k) - \bar{K}(k+1)H(k+1)F(k+1,k). \tag{19}$$

Thus, in the steady state case, after the steady state time is reached in s periods, the resulting discrete time Steady State Periodic Kalman Filter has as follows:

Steady State Periodic Kalman Filter (SSPKF)

$$\begin{aligned} x(sp+k+1/sp+k+1) = {} & \bar{A}(k \bmod p+1, k \bmod p)\,x(sp+k/sp+k) \\ & +\bar{K}(k \bmod p+1)\,z(sp+k+1), \end{aligned} \tag{20}$$

for $k = 0, 1, \ldots$

Remark 1.1 1. The Steady State Periodic Kalman Filter (SSPKF) implementation requires the Kalman Filter (KF) implementation for $k = 0, \ldots, sp$ in order to calculate the estimate $x(sp/sp)$.

2. The steady state periodic prediction error variance $\bar{P}(k+1,k)$ is calculated by off-line solving the corresponding discrete time periodic Riccati equation (12) using techniques for solving the discrete time Riccati equation [1]-[2] and [6]- [8]. Then, the steady state periodic gain matrix $\bar{K}(k+1)$ and the corresponding matrix $\bar{A}(k+1,k)$ are calculated off-line using (15) and (19), respectively.

2 Proposed steady state Kalman filter for periodic models

In this section, a new approach for the steady state Kalman filter implementation for periodic models is presented.

Using equation (20) after the steady state time is reached in s periods and taking advantage of the periodicity of the gain matrix $\bar{K}(k+1)$ and the corresponding matrix $\bar{A}(k+1,k)$, we are able to calculate the state estimate per period time, as shown in appendix A:

$$x(sp+\nu p/sp+\nu p) = [\bar{A}_1^p]^\nu x(sp/sp) + \sum_{i=1}^{p}\sum_{j=1}^{\nu}[\bar{A}_1^p]^{\nu-j}\bar{A}_{i+1}^p\,\bar{K}(i)\,z(sp+jp-p+i), \quad (21)$$

for $\nu = 1, 2, \ldots,$ where we define

$$\bar{A}_j^i = \begin{cases} \bar{A}(i,i-1)\cdot\bar{A}(i-1,i-2)\cdots\bar{A}(j,j-1), & i \geq j\ , \\ I, & i < j. \end{cases} \quad (22)$$

Thinking in an analogous way as for time invariant systems (period $p = 1$) [3], we observe that if all eigenvalues of $F(k+1,k)$ during one period $F(1,0)$, $F(2,1)$, $\ldots$, $F(p,p-1)$ lie inside the unit circle, then all eigenvalues of $\bar{A}(1,0)$, $\bar{A}(2,1)$, $\ldots$, $\bar{A}(p,p-1)$ lie inside the unit circle. Moreover, if all the singular values of $\bar{A}(1,0)$, $\bar{A}(2,1)$, $\ldots$, $\bar{A}(p,p-1)$ lie inside the unit circle, then it follows that all eigenvalues of the product matrix

$$\bar{A}_1^p = \bar{A}(p,p-1)\cdot\bar{A}(p-1,p-2)\cdots\bar{A}(2,1)\cdot\bar{A}(1,0)$$

lie also inside the unit circle. Using the above comments and the results in [4], we are able to conclude the following important property: *if the spectral radius of $\bar{A}_1^p$ is less than 1, then the computed powers of $\bar{A}_1^p$ can be expected to converge to zero.* Thus, there exists some ν, such that:

$$\left[\bar{A}_1^p\right]^\nu \neq 0 \quad \text{and} \quad \left[\bar{A}_1^p\right]^{\nu+i} = 0, \quad i = 1, 2, \ldots \quad (23)$$

Using the above property, after some algebra, as shown in appendix B, the expression in (21) can be written as:

$$x(sp+\nu p+\mu p/sp+\nu p+\mu p) = \sum_{i=1}^{p}\sum_{j=\mu}^{\nu+\mu} c(i,j)\,z(sp+jp-p+i), \quad (24)$$

where there exists some ν as in (23) and

$$c(i,j) = \left[\bar{A}_1^p\right]^{\nu+\mu-j}\ \bar{A}_{i+1}^p\,\bar{K}(i), \quad (25)$$

for $i = 1, 2, \ldots, p$ and $j = \mu, \mu+1, \ldots, \mu+\nu$.

Note that, if we consider the transformation: $w = j - \mu + 1$, the expressions in (24) and (25) can be written as follows:

$$x(sp + \nu p + \mu p / sp + \nu p + \mu p) = \sum_{i=1}^{p} \sum_{w=1}^{\nu+1} c(i, w + \mu - 1)\, z(sp + wp + \mu p - 2p + i) \quad (26)$$

$$c(i, w) = \left[\bar{A}_1^p\right]^{\nu+\mu-(w+\mu-1)} \; \bar{A}_{i+1}^p \, \bar{K}(i) = \left[\bar{A}_1^p\right]^{\nu-w+1} \; \bar{A}_{i+1}^p \, \bar{K}(i) \quad (27)$$

Thus, we obtain the following estimates per period time (p lags):

$$x(sp + \nu p + \mu p / sp + \nu p + \mu p) = \sum_{i=1}^{p} \sum_{j=1}^{\nu+1} c(i, j)\, z(sp + jp + \mu p - 2p + i), \quad (28)$$

where there exists some ν as in (23) and

$$c(i, j) = \left[\bar{A}_1^p\right]^{\nu-j+1} \; \bar{A}_{i+1}^p \, \bar{K}(i), \quad (29)$$

for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, \nu+1$.

Moreover, using (28) and (29), as shown in appendix C, we are able to write analogous estimates concerning each time ϕ in a period time $(\phi = 1, 2, \ldots, p)$, leading to a generalization of the expression in (28):

$$x(sp + \nu p + \mu p + \phi / sp + \nu p + \mu p + \phi) = \sum_{i=1}^{p} \sum_{j=1}^{\nu+2} \hat{c}(i, j)\, z(sp + jp + \mu p - 2p + i), \quad (30)$$

for $1 \le \phi \le p$, where there exists some ν as in (23) and

$$\hat{c}(i, j) = \begin{cases} \bar{A}_1^\phi \left[\bar{A}_1^p\right]^{\nu-j+1} \bar{A}_{i+1}^p \bar{K}(i), & 1 \le i \le p, \quad 1 \le j \le \nu+1, \\ \bar{A}_{i+1}^\phi \bar{K}(i), & 1 \le i \le \phi, \quad j = \nu+2, \\ 0, & i > \phi, \quad j = \nu+2 \end{cases} \quad (31)$$

Furthermore, using the property of eigenvalues $\lambda_i(AB) = \lambda_i(BA)$ and working as in (23), we remark that, due to the computer accuracy, the computed powers of $\left[\bar{A}_1^i \, \bar{A}_{i+1}^p\right]$, $i = 1, 2, \ldots, p$ can be expected to converge to zero. Thus, the coefficients $\hat{c}(i, 1)$ are written

$$\hat{c}(i, 1) = \bar{A}_1^\phi \left[\bar{A}_1^p\right]^\nu \; \bar{A}_{i+1}^p \, \bar{K}(i) = \bar{A}_{i+1}^\phi \left[\bar{A}_1^i \, \bar{A}_{i+1}^p\right]^{\nu+1} \bar{K}(i) \to 0, \quad 1 \le \phi \le p.$$

Consequently, substituting in (30) we have

$$\begin{aligned} x(sp + \nu p + \mu p + \phi / sp + \nu p + \mu p + \phi) &= \sum_{i=1}^{p} [\hat{c}(i, 2) z(sp + \mu p + i) + \hat{c}(i, 3) z(sp + \mu p + p + i) + \\ &\qquad + \cdots + \hat{c}(i, \nu+2) z(sp + \mu p + \nu p + i)] \\ &= \sum_{i=1}^{p} \sum_{\tau=1}^{\nu+2} \hat{c}(i, \tau)\, z(sp + \mu p + \tau p - 2p + i), \end{aligned}$$

with

$$\hat{c}(i,\tau) = \begin{cases} \bar{A}_1^{\phi}\left[\bar{A}_1^{p}\right]^{\nu-\tau+1}\bar{A}_{i+1}^{p}\bar{K}(i), & 1 \le i \le p, \quad 2 \le \tau \le \nu+1, \\ \bar{A}_{i+1}^{\phi}\bar{K}(i), & 1 \le i \le \phi, \quad \tau = \nu+2, \\ 0, & i > \phi, \quad \tau = \nu+2. \end{cases}$$

Using the two last expressions, due to $j = \tau - 1$, the following Proposed Steady State Periodic Kalman Filter implementation is derived:

Proposed Steady State Periodic Kalman Filter (PSSPKF)

$$x(sp+\nu p+\mu p+\phi/sp+\nu p+\mu p+\phi) = \sum_{i=1}^{p}\sum_{j=1}^{\nu+1}\hat{c}(i,j)\,z(sp+\mu p+jp-p+i), \tag{32}$$

for $1 \le \phi \le p$, where there exists some ν as in (23) and

$$\hat{c}(i,j) = \begin{cases} \bar{A}_1^{\phi}\left[\bar{A}_1^{p}\right]^{\nu-j}\bar{A}_{i+1}^{p}\bar{K}(i), & 1 \le i \le p, \quad 1 \le j \le \nu, \\ \bar{A}_{i+1}^{\phi}\bar{K}(i), & 1 \le i \le \phi, \quad j = \nu+1, \\ 0, & i > \phi, \quad j = \nu+1. \end{cases} \tag{33}$$

Remark 2.1

1. There is no need for any previous estimates calculation. The calculation of the steady state estimate $x(L)$ at some time L requires the use of the subset of $p(\nu+1)$ previous time measurements.

2. The steady state periodic prediction error variance $\bar{P}(k+1,k)$ is calculated by off-line solving the corresponding discrete time periodic Riccati equation (12) using techniques for solving the discrete time Riccati equation [1]-[2] and [6]-[8]. Then, the steady state periodic gain matrix $\bar{K}(k+1)$ and the corresponding matrix $\bar{A}(k+1,k)$ are calculated off-line using (15) and (19), respectively. Finally, the quantities, ν in (23) and the coefficients $\hat{c}(i,j)$ in (33) are calculated off-line.

3. This implementation can be used to obtain estimates concerning each time ϕ in a period time $(\phi = 1,2,\ldots,p)$.

4. If the matrices $\bar{A}_j = \bar{A}(j,j-1)$, $j = \phi+1,\ldots,p$, are nonsingular, since

$$\begin{aligned} \bar{A}_1^{\phi} = I\cdot\bar{A}_1^{\phi} &= \underbrace{\left(\bar{A}_{\phi+1}\right)^{-1}\left(\bar{A}_{\phi+2}\right)^{-1}\cdots\left(\bar{A}_{p}\right)^{-1}\bar{A}_p\cdots\bar{A}_{\phi+2}\bar{A}_{\phi+1}}_{I}\,\bar{A}_1^{\phi} \\ &= \left(\bar{A}_{\phi+1}\right)^{-1}\left(\bar{A}_{\phi+2}\right)^{-1}\cdots\left(\bar{A}_{p}\right)^{-1}\bar{A}_1^{p}, \end{aligned}$$

the coefficients

$$\begin{aligned} \hat{c}(i,1) &= \bar{A}_1^{\phi}\left[\bar{A}_1^{p}\right]^{\nu}\bar{A}_{i+1}^{p}\bar{K}(i) \\ &= \left(\bar{A}_{\phi+1}\right)^{-1}\left(\bar{A}_{\phi+2}\right)^{-1}\cdots\left(\bar{A}_{p}\right)^{-1}\underbrace{\bar{A}_1^{p}\left[\bar{A}_1^{p}\right]^{\nu}}\bar{A}_{i+1}^{p}\bar{K}(i) \to 0, \end{aligned}$$

for $1 \le \phi \le p$. Consequently, using (30) and (31) we conclude (32) and (33).

5. The proposed algorithm is a generalization of the proposed algorithm for time invariant systems in [3]. In fact, in the special case for $p = 1$ and $\phi = 1$, we derive the proposed algorithm in [3].

3 Computational analysis and simulation results

The computational analysis is based on the analysis in [3]. The following result was used: scalar addition and multiplication operations are involved in matrix manipulation operations, which are needed for the implementation of SSPKF (equation (20)) and PSSPKF (equation (32)). We define t_{op} as the time needed to calculate the result of the operation (sum/product) of two scalar operands. Table 1 summarizes the calculation time of matrix operations needed for the implementation of the algorithms.

Table 1. Calculation Time of Matrix Operations

Matrix Operation	Multiplications	Additions	Calculation Time
$(n \times m) \cdot (m \times k)$	nmk	$n\,(m-1)\,k$	$(2nmk - nk)\,t_{op}$
$(n \times 1) + (n \times 1)$	$--$	n	nt_{op}
t_{op} is the time needed to calculate the sum/product of two scalar operations			

In order to compare the algorithms we assume that we compute the estimate value $x(L/L)$ of the state vector $x(L)$ at some time $L = sp + \nu p + \mu p$, where sp is the time the steady state solution is reached, ν as in (23) and $\mu \ge 1$. We define the *calculation time* of each algorithm (SSPKF and PSSPKF) as the calculation time required for the on-line calculations of each algorithm (the calculation time of all off-line calculations is not taken into account):

$$t_{SSPKF} = \left(t_{KF} sp + (2n^2 + 2nm - n)p(\nu + \mu)\right) t_{op} \tag{34}$$

$$t_{PSSPKF} = ((2nm - n)(\nu + 1)p + n(p\nu + p - 1))\, t_{op} \tag{35}$$

where t_{KF} is the per recursion calculation time needed for the implementation of the Kalman filter (equations (3)-(7)). The calculation time of the algorithms are summarized in Table 2. Note that cb(PSSPKF) is constant (it depends on the state vector dimension n, on the measurement vector dimension m, on the period p and on the off-line calculated ν; it does not depend on μ).

Table 2. Calculation Time of Algorithms

Algorithm	Calculation Time
SSPKF	$(sp\, t_{KF} + (2n^2 + 2nm - n)p(\nu + \mu))\, t_{op}$
PSSPKF	$((2nm - n)(\nu + 1)p + n(p\nu + p - 1))\, t_{op}$
t_{KF} is the per recursion calculation time of Kalman filter	

We define the *time improvement* from SSPKF to PSSPKF as:

$$ti_{(SSPKF/PSSPKF)} = \frac{t_{SSPKF}}{t_{PSSPKF}} = \frac{sp\, t_{KF} + (2n^2 + 2nm - n)p(\nu + \mu)}{(2nm - n)(\nu + 1)p + n(p\nu + p - 1)} \tag{36}$$

From (36) we conclude that PSSPKF is faster than SSPKF:

$$\begin{aligned} ti_{(SSPKF/PSSPKF)} &> \frac{(2n^2 + 2nm - n)(\nu + \mu)}{(2nm - n)(\nu + 1) + n(\nu + 1)} \\ &= \frac{(2n + 2m - 1)(\nu + \mu)}{2m(\nu + 1)} > 1 \end{aligned} \tag{37}$$

and

$$ti_{(SSPKF/PSSPKF)} \approx \frac{(n + m)(\nu + \mu)}{m\nu} \tag{38}$$

is an increasing function of μ. Thus the proposed algorithm is fast. The proposed algorithm has a convenient parallel structure: all products

$$\hat{c}(i, j)\, z(sp + \mu p + jp - p + i)$$

in (32) can be computed in parallel, due to the fact that all coefficients $\hat{c}(i, j)$ in (33) can be off-line calculated (before the filter's implementation).

We define the *parallelism speedup* of PSSPKF as the calculation time needed for the sequential implementation of PSSPKF divided by the calculation time needed for the parallel implementation of PSSPKF:

$$\begin{aligned} speedup_{(PSSPKF)} &= \frac{(2nm - n)(\nu + 1)p + n(p\nu + p - 1)}{(2nm - n) + n(p\nu + p - 1)} \\ &\approx \frac{2m\nu p + \nu p}{2m + \nu p} > 1 \end{aligned} \tag{39}$$

Thus the proposed algorithm possesses a very good parallelism speedup.

Simulation Example.

A simple scalar case ($n = 1$ and $m = 1$) with period $p = 2$ is assumed in the following example:

$$F(1, 0) = 0.8 \;\text{ and }\; F(2, 1) = 0.6, \quad H(1) = 1 \;\text{ and }\; H(2) = 2,$$

$$Q(0) = 2 \text{ and } Q(1) = 5, \quad R(1) = 1 \text{ and } R(2) = 2,$$

with initial conditions $x(0/-1) = x_0 = 0$ and $P(0/-1) = P_0 = 0$. The steady state periodic prediction error variance is:

$$\bar{P}(1/0) = 2.2922, \quad \bar{P}(2/1) = 5.2506$$

and $s = 4$.
The steady state periodic gain is:

$$\bar{K}(1) = 0.6962, \quad \bar{K}(2) = 0.4565$$

and

$$\bar{A}(1,0) = 0.2430, \quad \bar{A}(2,1) = 0.0522.$$

Finally

$$\bar{A}_1^2 = \bar{A}(2,1) \cdot \bar{A}(1,0) = 0.0127$$

and $\nu = 9$.

The state $x(k)$ as well as the calculated estimates using SSPKF and PSSPKF are plotted in Figure 1. The estimates using SSPKF and PSSPKF are very close to each other.

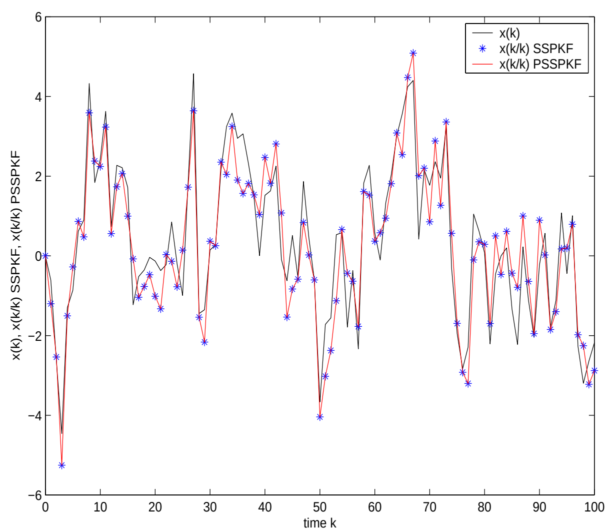

Figure 1: State $x(k)$ and estimates $x(k/k)$ using SSPKF and PSSPKF.

The proposed algorithm PSSPKF is faster than the classical algorithm SSPKF, as shown in Figure 2, where the time improvement from SSPKF to PSSPKF is plotted.

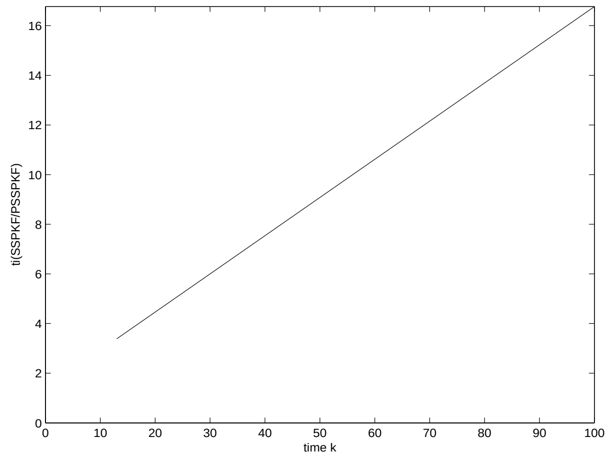

Figure 2: Time improvement from SSPKF to PSSPKF.

4 Conclusions

A new approach for the steady state Kalman filter is presented in this paper. The method is based on implementing the steady state Kalman filter equations in a different way than the classical algorithm does and taking advantage of the finite computer precision. The method requires the knowledge of a subset of previous time measurements in order to calculate the state estimate; there is no need of any previous estimates calculation. The classical and the proposed filters are equivalent with respect to their behavior. It was pointed out that the proposed algorithm is fast and has a parallel structure; this is very important due to the fact that, in most real-time applications, it is essential to obtain the estimate in the shortest possible time.

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Appendices

A Proof of equation (21)

Using equation (16):

$$x(k+1/k+1) = A(k+1,k)\,x(k/k) + K(k+1)\,z(k+1),$$

for $k = 0, 1, \ldots, p-1,$ we take

$$\begin{cases} x(1/1) = A(1,0)\,x(0/0) + K(1)\,z(1) \\ x(2/2) = A(2,1)\,x(1/1) + K(2)\,z(2) \\ \quad\vdots \\ x(p/p) = A(p,p-1)\,x(p-1/p-1) + K(p)\,z(p) \end{cases}$$

for $k = p, \ldots, p+(p-1) = 2p-1,$ we take

$$\begin{cases} x(p+1/p+1) = A(p+1,p)\,x(p/p) + K(p+1)\,z(p+1) \\ \quad\vdots \\ x(2p/2p) = A(2p,2p-1)\,x(2p-1/2p-1) + K(2p)\,z(2p) \end{cases}$$

and continuing for all the next k, we take

$$\begin{cases} x((s-1)p+1/(s-1)p+1) = A((s-1)p+1,(s-1)p)\,x((s-1)p/(s-1)p) \\ \qquad\qquad + K((s-1)p+1)\,z((s-1)p+1) \\ \quad\vdots \\ x(sp/sp) = A(sp,sp-1)\,x(sp-1/sp-1) + K(sp)\,z(sp) \end{cases}$$

Using the steady state expressions (20), and substituting in these the exactly previous time we have

$$\begin{cases} x(sp+1/sp+1) = \bar{A}(1,0)\,x(sp/sp) + \bar{K}(1)z(sp+1) \\ x(sp+2/sp+2) = \bar{A}(2,1)\,x(sp+1/sp+1) + \bar{K}(2)z(sp+2) \\ \qquad = \bar{A}(2,1)\bar{A}(1,0)x(sp/sp) + \bar{A}(2,1)\bar{K}(1)z(sp+1) + \bar{K}(2)z(sp+2) \\ x(sp+3/sp+3) = \bar{A}(3,2)\,x(sp+2/sp+2) + \bar{K}(3)z(sp+3) \\ \qquad = \bar{A}(3,2)\bar{A}(2,1)\bar{A}(1,0)x(sp/sp) + \bar{A}(3,2)\bar{A}(2,1)\bar{K}(1)z(sp+1) \\ \qquad\quad + \bar{A}(3,2)\bar{K}(2)z(sp+2) + \bar{K}(3)z(sp+3) \\ \quad\vdots \\ x(sp+p/sp+p) = \bar{A}(p,p-1)\,x(sp+p-1/sp+p-1) + \bar{K}(p)z(sp+p) \\ \qquad = \bar{A}(p,p-1)\bar{A}(p-1,p-2)\cdots\bar{A}(1,0)x(sp/sp) \\ \qquad\quad + \bar{A}(p,p-1)\bar{A}(p-1,p-2)\cdots\bar{A}(2,1)\bar{K}(1)z(sp+1) \\ \qquad\quad + \bar{A}(p,p-1)\bar{A}(p-1,p-2)\cdots\bar{A}(3,2)\bar{K}(2)z(sp+2) + \cdots \\ \qquad\quad + \bar{A}(p,p-1)\bar{K}(p-1)z(sp+p-1) + \bar{K}(p)z(sp+p) \end{cases}$$

By (22) the last expression is written

$$\begin{aligned} x(sp+p/sp+p) &= \bar{A}_1^p\,x(sp/sp) + \bar{A}_2^p\bar{K}(1)z(sp+1) + \bar{A}_3^p\bar{K}(2)z(sp+2) \\ &\qquad + \cdots + \bar{A}(p,p-1)\bar{K}(p-1)z(sp+p-1) + \bar{K}(p)z(sp+p) \\ &= \bar{A}_1^p x(sp/sp) + \sum_{i=1}^{p} \bar{A}_{i+1}^p \bar{K}(i)z(sp+i) \end{aligned}$$

Thus,

$$x(sp+p/sp+p) = \bar{A}_1^p x(sp/sp) + \sum_{i=1}^{p} \bar{A}_{i+1}^p \bar{K}(i)z(sp+i) \tag{A.1}$$

The next statement is constructed

$$\begin{cases} x(sp+p+1/sp+p+1) = \bar{A}(1,0)\,x(sp+p/sp+p) + \bar{K}(1)z(sp+p+1) \\ x(sp+p+2/sp+p+2) = \bar{A}(2,1)\,x(sp+p+1/sp+p+1) + \bar{K}(2)z(sp+p+2) \\ \qquad = \bar{A}(2,1)\bar{A}(1,0)x(sp+p/sp+p)+ \\ \qquad\quad + \bar{A}(2,1)\bar{K}(1)z(sp+p+1) + \bar{K}(2)z(sp+p+2) \\ \quad\vdots \end{cases}$$

$$\begin{cases} \quad\vdots \\ x(sp+p+p/sp+p+p) \\ \quad = \bar{A}(p,p-1)x(sp+p+p-1/sp+p+p-1) + \bar{K}(p)z(sp+p+p) \\ \quad = \bar{A}(p,p-1)\bar{A}(p-1,p-2)\cdots\bar{A}(1,0)x(sp+p/sp+p) \\ \qquad + \bar{A}(p,p-1)\bar{A}(p-1,p-2)\cdots\bar{A}(2,1)\bar{K}(1)z(sp+p+1) \\ \qquad + \bar{A}(p,p-1)\bar{A}(p-1,p-2)\cdots\bar{A}(3,2)\bar{K}(2)z(sp+p+2) + \cdots \\ \qquad + \bar{A}(p,p-1)\bar{K}(p-1)z(sp+p+p-1) + \bar{K}(p)z(sp+p+p) \\ \quad = \bar{A}_1^p x(sp+p/sp+p) + \bar{A}_2^p\bar{K}(1)z(sp+p+1) + \bar{A}_3^p\bar{K}(2)z(sp+p+2) \\ \qquad + \cdots + \bar{A}(p,p-1)\bar{K}(p-1)z(sp+p+p-1) + \bar{K}(p)z(sp+p+p) \end{cases}$$

The last expression can be written as

$$x(sp+p+p/sp+p+p) = \bar{A}_1^p x(sp+p/sp+p) + \sum_{i=1}^{p} \bar{A}_{i+1}^p \bar{K}(i)z(sp+p+i). \qquad \text{(A.2)}$$

Substituting (A.1) in (A.2) we have

$$\begin{aligned} & x(sp+2p/sp+2p) \\ & \quad = \bar{A}_1^p \left(\bar{A}_1^p x(sp/sp) + \sum_{i=1}^{p} \bar{A}_{i+1}^p \bar{K}(i)z(sp+i) \right) + \sum_{i=1}^{p} \bar{A}_{i+1}^p \bar{K}(i)z(sp+p+i) \\ & \quad = \left(\bar{A}_1^p\right)^2 x(sp/sp) + \sum_{i=1}^{p} \bar{A}_1^p \bar{A}_{i+1}^p \bar{K}(i)z(sp+i) + \sum_{i=1}^{p} \bar{A}_{i+1}^p \bar{K}(i)z(sp+p+i) \\ & \quad = \left(\bar{A}_1^p\right)^2 x(sp/sp) + \sum_{i=1}^{p} \left(\bar{A}_1^p \bar{A}_{i+1}^p \bar{K}(i)z(sp+i) + \left(\bar{A}_1^p\right)^0 \bar{A}_{i+1}^p \bar{K}(i)z(sp+p+i) \right) \\ & \quad = \left(\bar{A}_1^p\right)^2 x(sp/sp) + \sum_{i=1}^{p} \left(\sum_{r=0}^{1} \left(\bar{A}_1^p\right)^{1-r} \bar{A}_{i+1}^p \bar{K}(i)z(sp+rp+i) \right). \end{aligned}$$

Thus,

$$\begin{aligned} x(sp+2p/sp+2p) = {} & \left(\bar{A}_1^p\right)^2 x(sp/sp) \\ & + \sum_{i=1}^{p} \left(\sum_{r=0}^{1} \left(\bar{A}_1^p\right)^{1-r} \bar{A}_{i+1}^p \bar{K}(i)z(sp+rp+i) \right). \end{aligned} \qquad \text{(A.3)}$$

Continuing with similar way,

$$\begin{cases} x(sp+2p+1/sp+2p+1) = \bar{A}(1,0)\,x(sp+2p/sp+2p) + \bar{K}(1)z(sp+2p+1) \\ x(sp+2p+2/sp+2p+2) = \bar{A}(2,1)\,x(sp+2p+1/sp+2p+1) \\ \qquad\qquad + \bar{K}(2)z(sp+2p+2) \\ \quad\vdots \\ x(sp+2p+p/sp+2p+p) = \bar{A}(p,p-1)x(sp+2p+p-1/sp+2p+p-1) \\ \qquad\qquad + \bar{K}(p)z(sp+2p+p) \end{cases}$$

Thus,

$$x(sp+2p+p/sp+2p+p)$$
$$=\bar{A}(p,p-1)x(sp+2p+p-1/sp+2p+p-1)+\bar{K}(p)z(sp+2p+p) \quad \text{(A.4)}$$

and substituting in (A.4) the p previous statements, we take

$$x(sp+3p/sp+3p)=\bar{A}_1^p\,x(sp+2p/sp+2p)+\sum_{i=1}^{p}\bar{A}_{i+1}^p\bar{K}(i)z(sp+2p+i) \quad \text{(A.5)}$$

Then (A.5) can be written by (A.3)

$$\begin{aligned}
&x(sp+3p/sp+3p)\\
&\quad=\bar{A}_1^p\left(\left(\bar{A}_1^p\right)^2 x(sp/sp)+\sum_{i=1}^{p}\left(\sum_{r=0}^{1}\left(\bar{A}_1^p\right)^{1-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i)\right)\right)\\
&\qquad+\sum_{i=1}^{p}\bar{A}_{i+1}^p\bar{K}(i)z(sp+2p+i)\\
&\quad=\left(\bar{A}_1^p\right)^3 x(sp/sp)+\sum_{i=1}^{p}\bar{A}_1^p\left(\sum_{r=0}^{1}\left(\bar{A}_1^p\right)^{1-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i)\right)\\
&\qquad+\sum_{i=1}^{p}\bar{A}_{i+1}^p\bar{K}(i)z(sp+2p+i)\\
&\quad=\left(\bar{A}_1^p\right)^3 x(sp/sp)\\
&\qquad+\sum_{i=1}^{p}\left(\sum_{r=0}^{1}\left(\bar{A}_1^p\left(\bar{A}_1^p\right)^{1-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i)\right)+\left(\bar{A}_1^p\right)^0\bar{A}_{i+1}^p\bar{K}(i)z(sp+2p+i)\right).
\end{aligned}$$

Thus,

$$x(sp+3p/sp+3p)=\left(\bar{A}_1^p\right)^3 x(sp/sp)+\sum_{i=1}^{p}\left(\sum_{r=0}^{2}\left(\bar{A}_2^p\right)^{2-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i)\right).$$

Following the same procedure, we conclude

$$x(sp+\nu p/sp+\nu p)=\left(\bar{A}_1^p\right)^\nu x(sp/sp)+\sum_{i=1}^{p}\left(\sum_{r=0}^{\nu-1}\left(\bar{A}_2^p\right)^{\nu-1-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i)\right),$$

and in the last expression putting $r=j-1$, we lead to (21).

B Proof of equation (24)

Denoting $m = sp + \nu p$, and by (20), we take

$$\begin{cases} x(m+1/m+1) = \bar{A}(1,0)\,x(m/m) + \bar{K}(1)z(m+1) \\ x(m+2/m+2) = \bar{A}(2,1)\,x(m+1/m+1) + \bar{K}(2)z(m+2) \\ \qquad = \bar{A}(2,1)\bar{A}(1,0)x(m/m) + \bar{A}(2,1)\bar{K}(1)z(m+1) + \bar{K}(2)z(m+2) \\ \quad \vdots \\ x(m+p/m+p) = \bar{A}(p,p-1)\bar{A}(p-1,p-2)\cdots\bar{A}(2,1)\bar{A}(1,0)x(m/m) \\ \qquad + \bar{A}(p,p-1)\bar{A}(p-1,p-2)\cdots\bar{A}(2,1)\bar{K}(1)z(m+1) \\ \qquad + \cdots + \bar{A}(p,p-1)\,\bar{K}(p-1)z(m+p-1) + \bar{K}(p)z(m+p) \\ \qquad = \bar{A}_1^p\,x(m/m) + \bar{A}_2^p\,\bar{K}(1)z(m+1) + \bar{A}_3^p\,\bar{K}(2)z(m+2) \\ \qquad + \cdots + \bar{A}(p,p-1)\bar{K}(p-1)z(m+p-1) + \bar{K}(p)z(m+p) \\ \qquad = \bar{A}_1^p\,x(m/m) + \sum_{i=1}^{p} \bar{A}_{i+1}^p\bar{K}(i)z(m+i). \end{cases}$$

Thus,

$$x(m+p/m+p) = \bar{A}_1^p x(m/m) + \sum_{i=1}^{p} \bar{A}_{i+1}^p \bar{K}(i)z(m+i). \tag{B.1}$$

In (B.1), by (21) we take

$$\begin{aligned} &x(sp+\nu p+p/sp+\nu p+p) \\ &\quad = \bar{A}_1^p\,x(sp+\nu p/sp+\nu p) + \sum_{i=1}^{p} \bar{A}_{i+1}^p\bar{K}(i)z(sp+\nu p+i) \\ &\quad = \bar{A}_1^p\left(\left(\bar{A}_1^p\right)^{\nu} x(sp/sp) + \sum_{i=1}^{p}\left(\sum_{r=0}^{\nu-1}\left(\bar{A}_1^p\right)^{\nu-1-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i)\right)\right) \\ &\qquad\quad + \sum_{i=1}^{p}\bar{A}_{i+1}^p\bar{K}(i)z(sp+\nu p+i) \\ &\quad = \left(\bar{A}_1^p\right)^{\nu+1} x(sp/sp) + \sum_{i=1}^{p}\left(\sum_{r=0}^{\nu-1}\left(\bar{A}_1^p\right)^{\nu-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i)\right) \\ &\qquad\quad + \sum_{i=1}^{p}\bar{A}_{i+1}^p\bar{K}(i)z(sp+\nu p+i) \\ &\quad = \underbrace{\left(\bar{A}_1^p\right)^{\nu+1}}_{O} x(sp/sp) \\ &\qquad + \sum_{i=1}^{p}\left(\sum_{r=0}^{\nu-1}\left(\bar{A}_1^p\right)^{\nu-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i) + \bar{A}_{i+1}^p\bar{K}(i)z(sp+\nu p+i)\right) \\ &\quad = \sum_{i=1}^{p}\sum_{r=0}^{\nu}\left(\bar{A}_1^p\right)^{\nu-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i). \end{aligned}$$

Thus,

$$x(sp+\nu p+p/sp+\nu p+p)=\sum_{i=1}^{p}\sum_{r=0}^{\nu}\left(\bar{A}_1^p\right)^{\nu-r}\bar{A}_{i+1}^p\,\bar{K}(i)z(sp+rp+i). \qquad \text{(B.2)}$$

Moreover,

$$\begin{cases} x(m+p+1/m+p+1)=\bar{A}(1,0)\,x(m+p/m+p)+\bar{K}(1)z(m+p+1)\\ x(m+p+2/m+p+2)=\bar{A}(2,1)\,x(m+p+1/m+p+1)+\bar{K}(2)z(m+p+2)\\ \quad =\bar{A}(2,1)\bar{A}(1,0)x(m+p/m+p)+\bar{A}(2,1)\bar{K}(1)z(m+p+1)\bar{K}(2)z(m+p+2)\\ \vdots\\ x(m+2p/m+2p)=\bar{A}(p,p-1)\bar{A}(p-1,p-2)\cdots\bar{A}(2,1)\bar{A}(1,0)x(m+p/m+p)\\ \qquad +\bar{A}(p,p-1)\bar{A}(p-1,p-2)\cdots\bar{A}(2,1)\bar{K}(1)z(m+p+1)\\ \qquad +\cdots+\bar{A}(p,p-1)\,\bar{K}(p-1)z(m+2p-1)+\bar{K}(p)z(m+2p)\\ \\ \qquad =\bar{A}_1^p\,x(m+p/m+p)+\sum_{i=1}^{p}\bar{A}_{i+1}^p\bar{K}(i)z(m+p+i). \end{cases}$$

Thus,

$$x(m+2p/m+2p)=\bar{A}_1^p\,x(m+p/m+p)+\sum_{i=1}^{p}\bar{A}_{i+1}^p\,\bar{K}(i)z(m+p+i). \qquad \text{(B.3)}$$

Substituting (B.2) in (B.3) we take

$$\begin{aligned} &x(sp+\nu p+2p/sp+\nu p+2p)\\ &\quad=\bar{A}_1^p\left(\sum_{i=1}^{p}\sum_{r=0}^{\nu}\left(\bar{A}_1^p\right)^{\nu-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i)\right)+\sum_{i=1}^{p}\bar{A}_{i+1}^p\bar{K}(i)z(sp+\nu p+p+i)\\ &\quad=\sum_{i=1}^{p}\sum_{r=0}^{\nu+1}\left(\bar{A}_1^p\right)^{\nu+1-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i)\\ &\quad=\sum_{i=1}^{p}\left(\underbrace{\left(\bar{A}_1^p\right)^{\nu+1}}_{O}\bar{A}_{i+1}^p\bar{K}(i)z(sp+i)+\sum_{r=1}^{\nu+1}\left(\bar{A}_1^p\right)^{\nu+1-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i)\right)\\ &\quad=\sum_{i=1}^{p}\sum_{r=1}^{\nu+1}\left(\bar{A}_1^p\right)^{\nu+1-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i). \end{aligned}$$

Thus,

$$x(sp+\nu p+2p/sp+\nu p+2p)=\sum_{i=1}^{p}\sum_{r=1}^{\nu+1}\left(\bar{A}_1^p\right)^{\nu+1-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i). \qquad \text{(B.4)}$$

Since,

$$\begin{aligned} &x(sp+\nu p+3p/sp+\nu p+3p)\\ &\qquad=\bar{A}_1^p\,x(sp+\nu p+2p/sp+\nu p+2p)+\sum_{i=1}^{p}\bar{A}_{i+1}^p\,\bar{K}(i)z(sp+\nu p+2p+i), \end{aligned}$$

by (B.4) we have

$$\begin{aligned}
&x(sp+\nu p+3p/sp+\nu p+3p)\\
&=\bar{A}_1^p\left(\sum_{i=1}^{p}\sum_{r=1}^{\nu+1}\left(\bar{A}_1^p\right)^{\nu+1-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i)\right)+\sum_{i=1}^{p}\bar{A}_{i+1}^p\bar{K}(i)z(sp+\nu p+2p+i)\\
&=\sum_{i=1}^{p}\sum_{r=1}^{\nu+2}\left(\bar{A}_1^p\right)^{\nu+2-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i)\\
&\quad=\sum_{i=1}^{p}[\underbrace{\left(\bar{A}_1^p\right)^{\nu+1}}_{O}\bar{A}_{i+1}^p\bar{K}(i)z(sp+p+i)+\sum_{r=2}^{\nu+2}\left(\bar{A}_1^p\right)^{\nu+2-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i)]\\
&=\sum_{i=1}^{p}\sum_{r=2}^{\nu+2}\left(\bar{A}_1^p\right)^{\nu+2-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i).
\end{aligned}$$

Continuing with the same method, we construct

$$\begin{aligned}
&x(sp+\nu p+\mu p/sp+\nu p+\mu p)=\\
&\qquad=\sum_{i=1}^{p}\sum_{r=\mu-1}^{\nu+\mu-1}\left(\bar{A}_1^p\right)^{\nu+\mu-1-r}\bar{A}_{i+1}^p\bar{K}(i)z(sp+rp+i),
\end{aligned}\tag{B.5}$$

and by (B.5) for $j=r+1$, we obtain (24).

C Proof of equation (30)

We denote $t=sp+\nu p+\mu p$, and by (20) and (27), we take

$$\begin{cases}
x(t+1/t+1)=\bar{A}(1,0)\,x(t/t)+\bar{K}(1)z(t+1)\\
x(t+2/t+2)=\bar{A}(2,1)\,x(t+1/t+1)+\bar{K}(2)z(t+2)\\
\qquad\qquad=\bar{A}(2,1)\bar{A}(1,0)x(t/t)+\bar{A}(2,1)\bar{K}(1)z(t+1)+\bar{K}(2)z(t+2)\\
\quad\vdots\\
x(t+\phi/t+\phi)=\bar{A}(\phi,\phi-1)\bar{A}(\phi-1,\phi-2)\cdots\bar{A}(2,1)\bar{A}(1,0)x(t/t)\\
\qquad\qquad+\bar{A}(\phi,\phi-1)\bar{A}(\phi-1,\phi-2)\cdots\bar{A}(2,1)\bar{K}(1)z(t+1)\\
\qquad\qquad+\cdots+\bar{A}(\phi,\phi-1)\,\bar{K}(\phi-1)z(t+\phi-1)+\bar{K}(\phi)z(t+\phi)\\
\\
\qquad\qquad=\bar{A}_1^\phi\,x(t/t)+\sum_{i=1}^{\phi}\bar{A}_{i+1}^\phi\bar{K}(i)z(t+i).
\end{cases}$$

Thus,

$$x(t+\phi/t+\phi)=\bar{A}_1^\phi\,x(t/t)+\sum_{i=1}^{\phi}\bar{A}_{i+1}^\phi\bar{K}(i)z(t+i),\tag{C.1}$$

with $1\le\phi\le p$.

Then by (28), (29), we conclude

$$x(t+\phi/t+\phi) = \bar{A}_1^{\phi} x(t/t) + \sum_{i=1}^{\phi} \bar{A}_{i+1}^{\phi} \bar{K}(i) z(t+i)$$

$$= \bar{A}_1^{\phi} \sum_{i=1}^{p} \sum_{j=1}^{\nu+1} c(i,j) z(sp+jp+\mu p-2p+i) + \sum_{i=1}^{\phi} \bar{A}_{i+1}^{\phi} \bar{K}(i) z(t+i)$$

$$= \sum_{i=1}^{p} \bar{A}_1^{\phi} \sum_{j=1}^{\nu+1} c(i,j) z(sp+jp+\mu p-2p+i) + \sum_{i=1}^{p} d(i) z(t+i).$$

Thus,

$$x(t+\phi/t+\phi) = \sum_{i=1}^{p} \bar{A}_1^{\phi} \sum_{j=1}^{\nu+1} c(i,j) z(sp+jp+\mu p-2p+i) + \sum_{i=1}^{p} d(i) z(t+i), \quad \text{(C.2)}$$

where

$$d(i) = \begin{cases} \bar{A}_{i+1}^{\phi} \bar{K}(i), & 1 \le i \le \phi, \\ 0, & i > \phi, \end{cases} \qquad \phi = 1, 2, \ldots p. \quad \text{(C.3)}$$

Using (29) in (C.2), we obtain (30):

$$x(t+\phi/t+\phi) = \sum_{i=1}^{p} \bar{A}_1^{\phi} [\sum_{j=1}^{\nu+1} c(i,j) z(sp+jp+\mu p-2p+i)] + d(i) z(t+i)$$

$$= \sum_{i=1}^{p} \bar{A}_1^{\phi} [c(i,1) z(sp+\mu p-p+i) + c(i,2) z(sp+\mu p+i) + + \cdots + c(i,\nu+1) z(sp+\nu p+\mu p-p+i)] + d(i) z(t+i)$$

$$= \sum_{i=1}^{p} (\bar{A}_1^{\phi} [\bar{A}_1^{p}]^{\nu} \bar{A}_{i+1}^{p} \bar{K}(i) z(sp+\mu p-p+i) + \bar{A}_1^{\phi} [\bar{A}_1^{p}]^{\nu-1} \bar{A}_{i+1}^{p} \bar{K}(i) z(sp+\mu p+i) + \bar{A}_1^{\phi} [\bar{A}_1^{p}] \bar{A}_{i+1}^{p} \bar{K}(i) z(sp+\nu p+\mu p-2p+i) + \bar{A}_1^{\phi} \bar{A}_{i+1}^{p} \bar{K}(i) z(sp+\nu p+\mu p-p+i) + d(i) z(sp+\nu p+\mu p+i))$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{\nu+2} \hat{c}(i,j)\, z(sp+\mu p+jp-2p+i),$$

with

$$\hat{c}(i,j) = \begin{cases} \bar{A}_1^{\phi} \left[\bar{A}_1^{p}\right]^{\nu-j+1} \bar{A}_{i+1}^{p} \bar{K}(i), & 1 \le i \le p, & 1 \le j \le \nu+1, \\ d(i), & 1 \le i, & j = \nu+2, \end{cases} \qquad \phi = 1, 2, \ldots, p.$$

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