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LIMITED APPROXIMATION OF NUMERICAL RANGE OF NORMAL MATRIX

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Abstract. Let A be an $n \times n$ normal matrix, whose numerical range $NR[A]$ is a k-polygon. If a unit vector $v \in W \subseteq \mathbb{C}^n$, with $dimW = k$ and the point $v^*Av \in \mathrm{Int}NR[A]$, then $NR[A]$ is circumscribed to $NR[P^*AP]$, where P is an $n \times (k-1)$ isometry of $\{\mathrm{span}\{v\}\}^{\perp}_{W} \to \mathbb{C}^n$, [1]. In this paper, we investigate an internal approximation of $NR[A]$ by an increasing sequence of $NR[C_s]$ of compressed matrices $C_s = R_s^*AR_s$, with $R_s^*R_s = I_{k+s-1}$, $s = 1,2,\dots,n-k$ and additionally $NR[A]$ is expressed as limit of numerical ranges of k-compressions of A.

1. Introduction and preliminaries

Let $\mathscr{M}_n$ denote the algebra of all $n \times n$ complex matrices. The *numerical range* of $A \in \mathscr{M}_n$ is the well known set

$$NR[A] = \{x^*Ax \in \mathbb{C} : \ x \in \mathbb{C}^n \ \text{ with } \ \|x\|_2 = 1\},$$

which is a nonempty compact and convex subset of $\mathbb{C}$ that contains the spectrum $\sigma(A)$ of A (see [5, Chapter 1]). We recall that the numerical ranges of unitarily similar matrices are identified and if $A = MDM^*$; $D = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ is the unitary diagonalising form of a normal matrix A, then $NR[A] = \mathrm{Co}\{\sigma(A)\}$, where $\mathrm{Co}\{\cdot\}$ denotes the convex hull of the set.

Given two matrices $A \in \mathscr{M}_n$ and $C \in \mathscr{M}_k$ with $1 \leqslant k < n$, the matrix C is a *k-compression* of A, if there exists an $n \times k$ orthonormal matrix P (i.e., $P^*P = I_k$) such that $C = P^*AP$. Clearly,

$$NR[C] = NR[P^*AP] \subseteq NR[A], \tag{1}$$

and the equality holds only for $k = n$.

Moreover, we have

$$NR[C] \subseteq NR[PCP^*],$$

since $NR[C] = NR[P^*(PCP^*)P] \subseteq NR[PCP^*]$.

The numerical range of compressions of normal matrices have attracted attention and several results have been published in [1, 2, 3, 4]. The inclusion relation of $NR[A]$

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in (1) has been presented in details in [1], where the investigation leads to a structure of P such that the boundary of $NR[P^*AP]$ is supported by the edges of the boundary of $NR[A]$.

To explain, consider for a normal matrix $A \in \mathscr{M}_n$ the convex polygon $\mathscr{P} = \langle\lambda_1,\lambda_2,\ldots,\lambda_k\rangle = \mathrm{Co}\{\sigma(D)\} = NR[A]$, where the eigenvalues $\lambda_i, i = 1,\ldots,k,$ are simple. If $W = \mathrm{span}\{e_1,\ldots,e_k\}$ and $e_i,\ i = 1,\ldots,k,$ are vectors of the standard basis of $\mathbb{C}^n$, then for every unit vector

$$v = \sum_{i=1}^{k} \upsilon_i e_i \in W \quad ; \qquad \upsilon_i \in \mathbb{C}\backslash\{0\}, \quad i = 1,2,\ldots,k, \tag{2}$$

the point v^*Dv lies inside of the polygon $\mathscr{P}$. Denoting by $E_W^{\perp}(v)$ the orthogonal complement of $\mathrm{span}\{v\}$ with respect to the subspace W, clearly for the vector $\gamma = \gamma_1 e_1 + \cdots + \gamma_k e_k \in E_W^{\perp}(v)$, we have $\gamma \circ v = \sum_{i=1}^{k} \overline{\upsilon}_i\gamma_i = 0$ and further we take:

$$\gamma = \gamma_1 \left(e_1 - \frac{\overline{\upsilon}_1}{\overline{\upsilon}_j} e_j\right) + \gamma_2 \left(e_2 - \frac{\overline{\upsilon}_2}{\overline{\upsilon}_j} e_j\right) + \cdots + \gamma_k \left(e_k - \frac{\overline{\upsilon}_k}{\overline{\upsilon}_j} e_j\right), \tag{3}$$

for an index j. Therefore, by the vectors

$$\begin{aligned} b_1 &= e_1 - \frac{\overline{\upsilon}_1}{\overline{\upsilon}_j} e_j,\ \ldots,\ b_{j-1} = e_{j-1} - \frac{\overline{\upsilon}_{j-1}}{\overline{\upsilon}_j} e_j, \\ b_j &= e_{j+1} - \frac{\overline{\upsilon}_{j+1}}{\overline{\upsilon}_j} e_j,\ \ldots,\ b_{k-1} = e_k - \frac{\overline{\upsilon}_k}{\overline{\upsilon}_j} e_j, \end{aligned}$$

an orthonormal basis $\{w_1,w_2,\ldots,w_{k-1}\}$ of $E_W^{\perp}(v)$ is constructed. Defining the $n \times (k-1)$ matrix

$$P = \begin{bmatrix} w_1 & w_2 & \cdots & w_{k-1} \end{bmatrix}, \tag{4}$$

and $C = P^*DP$ the corresponding $(k-1)$-compression of $D = \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$, we conclude

$$\begin{aligned} NR[C] &= \{(Pz)^*D(Pz) : z \in \mathbb{C}^{k-1}, \|z\|_2 = 1\} = \{x^*Dx : x = Pz \in E_W^{\perp}(v),\ \|x\|_2 = 1\} \\ &\subseteq \{x^*Dx : x \in W,\ \|x\|_2 = 1\} = \langle\lambda_1,\lambda_2,\ldots,\lambda_k\rangle. \end{aligned}$$

Moreover, the following unit vectors of $\mathbb{C}^n \cap E_W^{\perp}(v)$

$$y_i = \frac{\overline{\upsilon}_{i+1}}{\sqrt{|\upsilon_i|^2 + |\upsilon_{i+1}|^2}} e_i - \frac{\overline{\upsilon}_i}{\sqrt{|\upsilon_i|^2 + |\upsilon_{i+1}|^2}} e_{i+1} \quad ; \quad i = 1,2,\ldots,k, \tag{5}$$

where in (5) e_{k+1} is substituted by e_1 and υ_{k+1} by υ_1, correspond to the points

$$c_i = y_i^*Dy_i = \frac{|\upsilon_{i+1}|^2\lambda_i + |\upsilon_i|^2\lambda_{i+1}}{|\upsilon_i|^2 + |\upsilon_{i+1}|^2} \quad ; \quad i = 1,2,\ldots,k \ ; \ \lambda_{k+1} = \lambda_1 \tag{6}$$

which belong to the line segment $\langle\lambda_i,\lambda_{i+1}\rangle \subset \partial NR[D]$. Obviously, the points c_i depend on the unit vector v and by Theorem 1 in [1] we have $\partial NR[D] \cap \partial NR[C] = \{c_1,\ldots,c_k\}$.

In the next section, we construct a sequence C_s, $s = 1,2,\dots,n-k$ of compressions of a normal matrix A such that the area of $NR[C_s]$ is increasing and is close enough to the polygon $\mathscr{P}$. Also, for $i = 1,2,\dots,k$, a sequence of points $t_{i,m} \in NR[C_{1,m}] \cap \langle \lambda_i, \lambda_r \rangle$ is constructed, where the matrix $C_{1,m}$ is a k-compression of A, depending on a vector ζ_m, with $\|\zeta_m\|_2 \to \infty$ and λ_r is an interior eigenvalue of the polygon, finding out that $\lim_{m\to\infty} t_{i,m} = \lambda_i$. By this statement we are led to $\lim_{m\to\infty} NR[C_{1,m}] = NR[A]$. Analogue results are obtained for subpolygons of $\mathscr{P}$.

2. An interior approximation of $NR[A]$

The interior approximation of the boundary of $NR[A]$ can be further elaborated, using a *compression* of a normal matrix A by a sequence of numerical ranges of suitable matrices.

PROPOSITION 1. *Let A be an $n\times n$ normal matrix, where its numerical range is a k-polygon $\mathscr{P}$. Then there exists a finite sequence of compressions $C_s = R_s^* D R_s$, with $R_s^* R_s = I_{k+s-1}$, $s = 1,2,\dots,n-k$, such that*

$$NR[C] \subseteq NR[C_1] \subseteq NR[C_2] \subseteq \cdots \subseteq NR[C_{n-k}] \subseteq NR[A], \tag{7}$$

and for every s, we have $\{c_1,\dots,c_k\} \subseteq NR[C_s] \cap \mathscr{P}$, with c_i in (6).

Proof. Consider the unit vector $v \in W$ in (2) and let

$$\xi_1 = v + \pi_{k+1} e_{k+1} = \sum_{i=1}^{k} \upsilon_i e_i + \pi_{k+1} e_{k+1}.$$

If $W_1 = \operatorname{span}\{W, e_{k+1}\}$ and $\gamma = (\gamma_1,\dots,\gamma_k,\gamma_{k+1}) \in E_{W_1}^{\perp}(\xi_1)$, then, $\gamma \circ \xi_1 = \sum_{i=1}^{k} \overline{\upsilon}_i \gamma_i + \overline{\pi}_{k+1}\gamma_{k+1} = 0$ and for the same index j as in (3), we have:

$$\gamma = \gamma_1 \left(e_1 - \frac{\overline{\upsilon}_1}{\overline{\upsilon}_j} e_j \right) + \cdots + \gamma_k \left(e_k - \frac{\overline{\upsilon}_k}{\overline{\upsilon}_j} e_j \right) + \gamma_{k+1} \left(e_{k+1} - \frac{\overline{\pi}_{k+1}}{\overline{\upsilon}_j} e_j \right).$$

Thus, the orthonormal basis $\{w_1,\dots,w_{k-1},r_1\}$ is constructed by the vectors

$$b_1 = e_1 - \frac{\overline{\upsilon}_1}{\overline{\upsilon}_j} e_j, \dots, \quad b_{j-1} = e_{j-1} - \frac{\overline{\upsilon}_{j-1}}{\overline{\upsilon}_j} e_j,$$
$$b_j = e_{j+1} - \frac{\overline{\upsilon}_{j+1}}{\overline{\upsilon}_j} e_j, \dots, \quad b_{k-1} = e_k - \frac{\overline{\upsilon}_k}{\overline{\upsilon}_j} e_j, \quad b_k = e_{k+1} - \frac{\overline{\pi}_{k+1}}{\overline{\upsilon}_j} e_j.$$

Denoting by $R_1 = \begin{bmatrix} P & r_1 \end{bmatrix}$, where P is the matrix in (4), clearly $R_1^* R_1 = I_k$ and the equation

$$C_1 = R_1^* D R_1 = \begin{bmatrix} P^* D P & P^* D r_1 \\ r_1^* D P & r_1^* D r_1 \end{bmatrix} \tag{8}$$

yields the inclusion

$$NR[C] = NR[P^*DP] \subseteq NR[C_1].$$

If $W_2 = \text{span}\{W_1, e_{k+2}\} = \text{span}\{W, e_{k+1}, e_{k+2}\}$ and $\xi_2 = \xi_1 + \pi_{k+2}e_{k+2}$, similarly we define the orthonormal basis $\{w_1, \ldots, w_{k-1}, r_1, r_2\}$ of $E^{\perp}_{W_2}(\xi_2)$ and the matrix $R_2 = \begin{bmatrix} P & r_1 & r_2 \end{bmatrix} = \begin{bmatrix} R_1 & r_2 \end{bmatrix}$. Thus,

$$C_2 = R_2^* D R_2 = \begin{bmatrix} R_1^* D R_1 & R_1^* D r_2 \\ r_2^* D R_1 & r_2^* D r_2 \end{bmatrix},$$

concluding that $NR[C_1] = NR[R_1^* D R_1] \subseteq NR[C_2]$. Continuing in the same way, we consider the vector

$$\xi_{n-k} = v + \pi_{k+1}e_{k+1} + \pi_{k+2}e_{k+2} + \cdots + \pi_n e_n$$

of subspace $W_{n-k} = \text{span}\{W, e_{k+1}, \ldots, e_n\}$ and for the same index j as in (3), we receive the orthonormal basis $\{w_1, \ldots, w_{k-1}, r_1, r_2, \ldots, r_{n-k}\}$ of $E^{\perp}_{W_{n-k}}(\xi_{n-k})$. If

$$C_{n-k} = R_{n-k}^* D R_{n-k},$$

where $R_{n-k} = \begin{bmatrix} R_{n-k-1} & r_{n-k} \end{bmatrix} = \cdots = \begin{bmatrix} P & r_1 & r_2 & \cdots & r_{n-k} \end{bmatrix}_{n\times(n-1)}$, clearly

$$NR[C_{n-k-1}] \subseteq NR[C_{n-k}] \subseteq NR[A].$$

Furthermore, by the inclusions in (7), we have that the tangential points c_i in (6) of $NR[C]$ and the polygon $\mathscr{P} = NR[A]$, belong also to $NR[C_s]$, for $s = 1, \ldots, n-k$. Note that the vectors y_i in (5) belong to the subspaces $E^{\perp}_{W_s}(\xi_s)$, with $\xi_s = v + \pi_{k+1}e_{k+1} + \pi_{k+2}e_{k+2} + \cdots + \pi_{k+s}e_{k+s}$, $s = 1, 2, \ldots, n-k$ and for the unit vectors $g_{i,s}$, $i = 1, \ldots, k$, defined by the equation $R_s g_{i,s} = y_i$, we have:

$$c_i = \frac{|\upsilon_i|^2 \lambda_{i+1} + |\upsilon_{i+1}|^2 \lambda_i}{|\upsilon_i|^2 + |\upsilon_{i+1}|^2} = y_i^* D y_i = g_{i,s}^* (R_s^* D R_s) g_{i,s} = g_{i,s}^* C_s g_{i,s},$$

with $\|g_{i,s}\|_2 = 1$. □

If, instead of v in (2), we consider the vector

$$u = \sum_{j=k+1}^{n} u_j e_j,$$

where $e_{k+1}, \ldots, e_n$ are the remaining vectors of the standard basis of $\mathbb{C}^n$ and simultaneously the eigenvectors of D corresponding to the eigenvalues in the interior of polygon $\mathscr{P}$, then $E^{\perp}_W(u) = W$. Thus for $\widetilde{P} = \begin{bmatrix} e_1 & e_2 & \cdots & e_k \end{bmatrix}_{n\times k} = \begin{bmatrix} I_k \\ \mathbb{O}_{n-k} \end{bmatrix}$, we obtain:

$$\begin{aligned} NR[\widetilde{P}^* D \widetilde{P}] &= \{(\widetilde{P}z)^* D (\widetilde{P}z) : z \in \mathbb{C}^k, \|z\|_2 = 1\} \\ &= \{z^* \text{diag}(\lambda_1, \ldots, \lambda_k) z : z \in \mathbb{C}^k, \|z\|_2 = 1\} = \mathscr{P}. \end{aligned}$$

Regarding a vector,

$$\beta_{i,\tau} = u + \sum_{j=i}^{\tau} \rho_j e_j \; ; \; \text{ for } \; 1 \leqslant i \leqslant \tau \leqslant k, \tag{9}$$

along similar lines as in Proposition 1, we conclude the following proposition.

PROPOSITION 2. *Let A be an $n \times n$ normal matrix, whose the numerical range is a k-polygon $\mathscr{P}$. Let also a vector $\beta_{i,\tau}$ as in (9). Then there exists a $(n-1)$-compression $\widetilde{C}_{i,\tau}$ of D such that*

$$NR[\widetilde{C}_{i,\tau}] = Co\{\langle \lambda_1, \dots, \lambda_{i-1} \rangle \cup \langle \lambda_{\tau+1}, \dots, \lambda_k \rangle \cup NR[B_{i,\tau}]\}, \tag{10}$$

where $B_{i,\tau}$ is an $(n-k+\tau-i)$-compression of D.

Proof. Let a vector $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n) \in \operatorname{span}\{\beta_{i,\tau}\}^{\perp}$, then

$$\gamma \circ \beta_{i,\tau} = \sum_{j=k+1}^{n} \overline{u}_j \gamma_j + \sum_{j=i}^{\tau} \overline{\rho}_j \gamma_j = 0.$$

Thus, for an index ℓ with $k+1 \leqslant \ell \leqslant n$, we have:

$$\gamma = \gamma_1 e_1 + \dots + \gamma_i \left(e_i - \frac{\overline{\rho}_i}{\overline{u}_\ell} e_\ell \right) + \dots + \gamma_\tau \left(e_\tau - \frac{\overline{\rho}_\tau}{\overline{u}_\ell} e_\ell \right) + \gamma_{\tau+1} e_{\tau+1} + \dots + \gamma_k e_k$$
$$+ \gamma_{k+1} \left(e_{k+1} - \frac{\overline{u}_{k+1}}{\overline{u}_\ell} e_\ell \right) + \dots + \gamma_n \left(e_n - \frac{\overline{u}_n}{\overline{u}_\ell} e_\ell \right)$$

By the vectors $\omega_j = e_j - \frac{\overline{\rho}_j}{\overline{u}_\ell} e_\ell$ $(j = i, \dots, \tau)$ and $\phi_{k+1} = e_{k+1} - \frac{\overline{u}_{k+1}}{\overline{u}_\ell} e_\ell, \dots, \phi_{\ell-1} = e_{\ell-1} - \frac{\overline{u}_{\ell-1}}{\overline{u}_\ell} e_\ell, \phi_\ell = e_{\ell+1} - \frac{\overline{u}_{\ell+1}}{\overline{u}_\ell} e_\ell, \dots, \phi_{n-1} = e_n - \frac{\overline{u}_n}{\overline{u}_\ell} e_\ell$ an orthonormal basis

$$\{e_1, \dots, e_{i-1}, \hat{\omega}_i, \dots, \hat{\omega}_\tau, e_{\tau+1}, \dots, e_k, \hat{\phi}_{k+1}, \dots, \hat{\phi}_{n-1}\}$$

is constructed and the $n \times (n-1)$ matrix

$$\widetilde{P}_{i,\tau} = \begin{bmatrix} Q_1 & Q_2 & \Omega & \Phi \end{bmatrix}, \tag{11}$$

where $Q_1 = \begin{bmatrix} e_1 \, e_2 \cdots e_{i-1} \end{bmatrix}$, $Q_2 = \begin{bmatrix} e_{\tau+1} \, e_{\tau+2} \cdots e_k \end{bmatrix}_{n \times (k-\tau)}$, $\Omega = \begin{bmatrix} \hat{\omega}_i \cdots \hat{\omega}_\tau \end{bmatrix}_{n \times (\tau-i+1)}$, $\Phi = \begin{bmatrix} \hat{\phi}_{k+1} \cdots \hat{\phi}_{n-1} \end{bmatrix}_{n \times (n-k-1)}$, is an isometry. Hence by the $(n-1)$-compression of D

$$\widetilde{C}_{i,\tau} = \widetilde{P}_{i,\tau}^* D \widetilde{P}_{i,\tau} = \begin{bmatrix} Q_1^* D Q_1 & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & Q_2^* D Q_2 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \Omega^* D \Omega & \Omega^* D \Phi \\ \mathbb{O} & \mathbb{O} & \Phi^* D \Omega & \Phi^* D \Phi \end{bmatrix} \tag{12}$$
$$= \begin{bmatrix} \operatorname{diag}(\lambda_1, \dots, \lambda_{i-1}) & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \operatorname{diag}(\lambda_{\tau+1}, \dots, \lambda_k) & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \Omega^* D \Omega & \Omega^* D \Phi \\ \mathbb{O} & \mathbb{O} & \Phi^* D \Omega & \Phi^* D \Phi \end{bmatrix}$$

we are led to the relation

$$NR[\widetilde{C}_{i,\tau}] = \mathrm{Co}\{\langle\lambda_1,\ldots,\lambda_{i-1}\rangle \cup \langle\lambda_{\tau+1},\ldots,\lambda_k\rangle \cup NR[B_{i,\tau}]\},$$

where $B_{i,\tau} = \begin{bmatrix} \Omega^*D\Omega & \Omega^*D\Phi \\ \Phi^*D\Omega & \Phi^*D\Phi \end{bmatrix}$ is an $(n-k+\tau-i)$-compression of D. $\square$

Considering the vector

$$\beta_1 = u + \rho_i e_i\,;\;\; i \in \{1,2,\ldots,k\},$$

as in (9), by (12) we may construct the corresponding compression

$$\widetilde{C}_1 = \mathrm{diag}\,(\mathrm{diag}\,(\lambda_1,\ldots,\lambda_{i-1}), \mathrm{diag}\,(\lambda_{i+1},\ldots,\lambda_k), B_1)\,, \tag{13}$$

where $B_1 = \begin{bmatrix} \hat{\omega}_i^*D\hat{\omega}_i & \hat{\omega}_i^*D\Phi \\ \Phi^*D\hat{\omega}_i & \Phi^*D\Phi \end{bmatrix}$ is $(n-k)$-compression of D. Due to the construction of the orthonormal basis $\{\hat{\omega}_i,\Phi\}$, $\partial NR[B_1]$ is inscribed to the polygon $\langle\lambda_i,\lambda_{k+1},\ldots,\lambda_n\rangle$.

PROPOSITION 3. *Let A be an $n\times n$ normal matrix and the polygon $\mathscr{P}=\langle\lambda_1,\ldots,\lambda_k\rangle = NR[A]$.*

I. *If we consider a sequence of vectors $\zeta_m = v + q_m e_j$, where e_j is eigenvector of D corresponding to the interior eigenvalue λ_j of $\mathscr{P}$, such that $\lim_{m\to\infty}\|\zeta_m\|_2 = \infty$, and the matrices $C_{1,m} \in \mathscr{M}_k$ are the k-compressions of D in (8) defined by ζ_m, then there exists a sequence of points $t_{i,m} \in NR[C_{1,m}] \cap \langle\lambda_i,\lambda_j\rangle$, such that $\lim_{m\to\infty} t_{i,m} = \lambda_i$, for $i \in \{1,2,\ldots,k\}$.*

II. *Let $\beta_m = \sum_{j=k+1}^{n} u_{j,m}e_j + \rho_i e_i$ be a sequence of vectors, with $i \in \{1,2,\ldots,k\}$. If $\lim_{m\to\infty}|u_{j,m}| = \infty$ holds for a prefixed j, and $\widetilde{C}_{1,m}$ is the corresponding $(n-1)$-compression of D in (13), then there exists a sequence of points $c_{i,m} \in NR[\widetilde{C}_{1,m}] \cap \langle\lambda_i,\lambda_j\rangle$, such that $\lim_{m\to\infty} c_{i,m} = \lambda_i$.*

Proof. **I.** Consider the unit vectors of $\mathbb{C}^n$

$$z_{i,m} = \frac{\overline{q}_m}{\sqrt{|\upsilon_i|^2+|q_m|^2}}\,e_i - \frac{\overline{\upsilon}_i}{\sqrt{|\upsilon_i|^2+|q_m|^2}}\,e_j\,, \quad i = 1,2,\ldots,k,$$

and the vectors $f_{i,m} \in \mathbb{C}^k$ defined by the equations $\tilde{R}_{1,m}f_{i,m} = z_{i,m}$, where the $n\times k$ matrix $\tilde{R}_{1,m}$ is constructed by ζ_m, as R_1 in Proposition 1. Obviously, $\|f_{i,m}\|_2 = 1$ and the points

$$\begin{aligned} t_{i,m} &= f_{i,m}^*C_{1,m}f_{i,m} = f_{i,m}^*\tilde{R}_{1,m}^*D\tilde{R}_{1,m}f_{i,m} = z_{i,m}^*Dz_{i,m} = \frac{|\upsilon_i|^2\lambda_j + |q_m|^2\lambda_i}{|\upsilon_i|^2+|q_m|^2} \\ &= \lambda_i + \frac{|\upsilon_i|^2}{|\upsilon_i|^2+|q_m|^2}\,(\lambda_j-\lambda_i), \quad i = 1,2,\ldots,k, \end{aligned} \tag{14}$$

belong to $NR[C_{1,m}]\cap\langle\lambda_i,\lambda_j\rangle$. Moreover, due to $\lim_{m\to\infty}\|\zeta_m\|_2=\infty$ by (14) we have

$$\lim_{m\to\infty} t_{i,m} = \lim_{m\to\infty}(\lambda_i + \frac{|\upsilon_i|^2}{|\upsilon_i|^2+|q_m|^2}(\lambda_j-\lambda_i)) = \lambda_i.$$

II. Consider the unit vectors of $\mathbb{C}^n$

$$\psi_{j,m} = \frac{\overline{\rho}_i}{\sqrt{|u_{j,m}|^2+|\rho_i|^2}}e_j - \frac{\overline{u}_{j,m}}{\sqrt{|u_{j,m}|^2+|\rho_i|^2}}e_i,$$

and the vectors $h_{j,m}\in\mathbb{C}^{n-1}$ defined by the equations $\widetilde{P}_{1,m}h_{j,m}=\psi_{j,m}$, where $\widetilde{P}_{1,m}$ is constructed as in (11). Then, the point

$$c_{i,m} = h_{j,m}^*\widetilde{C}_{1,m}h_{j,m} = h_{j,m}^*\widetilde{P}_{1,m}^*D\widetilde{P}_{1,m}h_{j,m} = \psi_{j,m}^*D\psi_{j,m} = \frac{|u_{j,m}|^2\lambda_i+|\rho_i|^2\lambda_j}{|u_{j,m}|^2+|\rho_i|^2} \tag{15}$$

belongs to $NR[\widetilde{C}_{1,m}]\cap\langle\lambda_i,\lambda_j\rangle$, and due to $\lim_{m\to\infty}|u_{j,m}|=\infty$, the equality in (15) yields

$$\lim_{m\to\infty} c_{i,m} = \lim_{m\to\infty}(\lambda_i + \frac{|\rho_i|^2}{|u_{j,m}|^2+|\rho_i|^2}(\lambda_j-\lambda_i)) = \lambda_i. \qquad \square$$

Clearly, by Proposition 3 **I**, $\lim_{m\to\infty}|t_{i,m}|=|\lambda_i|$. If the point $\ell_{i,m}$ lies on $\partial NR[C_{1,m}]\cap\langle\lambda_i,\lambda_j\rangle$, we have

$$|t_{i,m}|\leqslant|\ell_{i,m}|\leqslant|\lambda_i|,$$

i.e.,

$$\lim_{m\to\infty}|\ell_{i,m}|=|\lambda_i|.$$

Therefore, there exist an index $m_0(i)\in\mathbb{N}$ and small enough $\varepsilon>0$, such that for $m\geqslant m_0(i)$, the distance $d(t_{i,m},\partial NR[C_{1,m}])<\varepsilon$.

Numerically, we may assume that the equality $|t_{i,m_0(i)}|\approx|\ell_{i,m_0(i)}|$ holds and the equality in (14) leads to

$$|t_{i,m}| = \frac{|\upsilon_i|^2}{|\upsilon_i|^2+|q_m|^2}|\lambda_j| + \frac{|q_m|^2}{|\upsilon_i|^2+|q_m|^2}|\lambda_i|,$$

whereby we derive

$$|q_{m_0(i)}|^2 \approx |\upsilon_i|^2\frac{|\ell_{i,m_0(i)}|-|\lambda_j|}{|\lambda_i|-|\ell_{i,m_0(i)}|}.$$

Moreover, for $m_1<m_2$, due to $\lim_{m\to\infty}\|\zeta_m\|_2=\infty$ by (14) we have

$$|t_{i,m_1}-\lambda_j| = \frac{|q_{m_1}|^2}{|\upsilon_i|^2+|q_{m_1}|^2}|\lambda_i-\lambda_j| \leqslant \frac{|q_{m_2}|^2}{|\upsilon_i|^2+|q_{m_2}|^2}|\lambda_i-\lambda_j| = |t_{i,m_2}-\lambda_j|,$$

yielding

$$NR[C_{1,m_1}] \subseteq NR[C_{1,m_2}]. \tag{16}$$

Since the sequence $|q_m|$ is increasing, by (16) for $m \geqslant m_0(i)$, we conclude

$$|q_m|^2 = |\upsilon_i|^2 \frac{|t_{i,m}| - |\lambda_j|}{|\lambda_i| - |t_{i,m}|} \geqslant |q_{m_0(i)}|^2 = |\upsilon_i|^2 \frac{|\ell_{i,m_0(i)}| - |\lambda_j|}{|\lambda_i| - |\ell_{i,m_0(i)}|},$$

i.e., $t_{i,m}$ has to be nearly a boundary point of $NR[C_{1,m}]$. Hence, for $m = m_0(i)$ we can write

$$t_{i,m_0(i)} \approx f^*_{i,m_0(i)} C_{1,m_0(i)} f_{i,m_0(i)},$$

where $f_{i,m_0(i)}$ is the eigenvector of $H(e^{-\mathbf{i}\theta_i} C_{1,m_0(i)})$ corresponding to the largest eigenvalue, $\lambda_{\max}(H(e^{-\mathbf{i}\theta_i} C_{1,m_0(i)}))$, of hermitian part of matrix $e^{-\mathbf{i}\theta_i} C_{1,m_0(i)}$, and $\theta_i \in [0, 2\pi)$ is the argument of $t_{i,m_0(i)}$, (see [5, p. 35, Theorem 1.5.11]).

THEOREM 4. *For any normal matrix A, whose $NR[A]$ is a k-polygon, there exists a sequence of k-compressions $C_{1,m}$ of D in (8) such that $NR[C_{1,m}]$ is inscribed to the polygon for every m and $\lim_{m\to\infty} NR[C_{1,m}] = NR[A]$.*

Proof. Let $\mathcal{Q}_m = \mathrm{Co}\{t_{1,m(1)}, \ldots, t_{k,m(k)}\}$. If $m_0 = \max\{m_0(1), m_0(2), \ldots, m_0(k)\}$, then by Proposition 3 **I**, for $m > m_0$ and small enough $\varepsilon > 0$, we estimate that

$$|\mathcal{Q}_m| \leqslant |NR[C_{1,m}]| \leqslant |\mathcal{P}|,$$

where $|\cdot|$ denotes the area of a convex set. Since $\lim_{m\to\infty} \mathcal{Q}_m = \mathcal{P}$, obviously we have the convergence of area of $NR[C_{1,m}]$ to the area contained in $NR[A]$. □

COROLLARY 5. *For any normal matrix A, whose $NR[A]$ is a k-polygon, there exists a sequence of vectors $\beta_m = \sum_{j=k+1}^{n} u_{j,m} e_j + \rho_i e_i$; $i \in \{1, 2, \ldots, k\}$, and the associated sequence $\widetilde{C}_{1,m}$ of $(n-1)$-compressions of D in (13), such that*

$$\lim_{m\to\infty} NR[\widetilde{C}_{1,m}] = NR[A],$$

when $\lim_{m\to\infty} |u_{j,m}| = \infty$.

Proof. Since, by (13) the compression matrix

$$\widetilde{C}_{1,m} = \mathrm{diag}(\mathrm{diag}(\lambda_1, \ldots, \lambda_{i-1}), \mathrm{diag}(\lambda_{i+1}, \ldots, \lambda_k), B_{1,m}),$$

clearly

$$NR[B_{1,m}] \subseteq \mathrm{Co}\{\lambda_i, \lambda_{k+1}, \ldots, \lambda_n\},$$

and due to $\lim_{m\to\infty} c_{i,m} = \lambda_i$, we obtain

$$\lim_{m\to\infty} NR[B_{1,m}] = \langle \lambda_i, \lambda_{k+1}, \ldots, \lambda_n \rangle.$$

Hence, by (10) we have

$$\begin{aligned}\lim_{m\to\infty} NR[\widetilde{C}_{1,m}] &= \mathrm{Co}\{\langle \lambda_1, \ldots, \lambda_{i-1}\rangle \cup \langle \lambda_{i+1}, \ldots, \lambda_k\rangle \cup \lim_{m\to\infty} NR[B_{1,m}]\} \\ &= \mathrm{Co}\{\langle \lambda_1, \ldots, \lambda_{i-1}\rangle \cup \langle \lambda_{i+1}, \ldots, \lambda_k\rangle \cup \langle \lambda_i, \lambda_{k+1}, \ldots, \lambda_n\rangle\} = \langle \lambda_1, \ldots, \lambda_k\rangle.\end{aligned}$$

□

The next example illustrates Proposition 3 **I** and indirectly Theorem 4.

EXAMPLE. Let the 6×6 normal matrix $A = \mathrm{diag}(4\mathbf{i}, -2, -3\mathbf{i}, 5, 0, 1+\mathbf{i})$, where $NR[A] = \mathrm{Co}\{4\mathbf{i}, -2, -3\mathbf{i}, 5\}$, i.e., 0 and $1+\mathbf{i}$ belong to $\mathrm{Int}\, NR[A]$. For the unit vector $v = \frac{1}{\sqrt{15}} e_1 + \frac{2}{\sqrt{15}} e_2 + \frac{1}{\sqrt{15}} e_3 + \frac{3}{\sqrt{15}} e_4$, we have the matrix in (4),

$$P = \begin{bmatrix} -0.8944 & -0.1826 & -0.3162 \\ 0.4472 & -0.3651 & -0.6325 \\ 0 & 0.9129 & -0.3162 \\ 0 & 0 & 0.6325 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the tangent points in (6) of $\partial NR[A] \cap \partial NR[P^*AP]$

$$c_1 = \frac{-2+16\mathbf{i}}{5}, \quad c_2 = \frac{-2-12\mathbf{i}}{5}, \quad c_3 = \frac{5-27\mathbf{i}}{10}, \quad c_4 = \frac{5+36\mathbf{i}}{10}.$$

If $\zeta_1 = v + \frac{4}{\sqrt{15}} e_5$, we obtain $\tilde{R}_{1,1} = \begin{bmatrix} -0.8944 & -0.1826 & -0.3162 & -0.1855 \\ 0.4472 & -0.3651 & -0.6325 & -0.3710 \\ 0 & 0.9129 & -0.3162 & -0.1855 \\ 0 & 0 & 0.6325 & -0.5565 \\ 0 & 0 & 0 & 0.6956 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and the matrix $C_{1,1} = \tilde{R}_{1,1}^* A \tilde{R}_{1,1}$ as in (8). By (14), for $\lambda_5 = 0$, we take: $t_{1,1} = \frac{64\mathbf{i}}{17} = 3.7647\mathbf{i} \in \langle \lambda_1, \lambda_5 \rangle$. Also, $\theta_1 = \pi/2$ and $|t_{1,1}| = 3.7647 \neq 3.8691 = \lambda_{\max}(H(e^{-\mathbf{i}\theta_1} C_{1,1}))$, i.e., $t_{1,1}$ is interior point of $NR[C_{1,1}]$.

Similarly, if $\zeta_2 = v + \frac{20}{\sqrt{15}} e_5$, we have $\tilde{R}_{1,2} = \begin{bmatrix} -0.8944 & -0.1826 & -0.3162 & -0.2535 \\ 0.4472 & -0.3651 & -0.6325 & -0.5070 \\ 0 & 0.9129 & -0.3162 & -0.2535 \\ 0 & 0 & 0.6325 & -0.7605 \\ 0 & 0 & 0 & 0.1901 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

and the matrix $C_{1,2} = \tilde{R}_{1,2}^* A \tilde{R}_{1,2}$ as in (8). By (14) we take: $t_{1,2} = \frac{1600\mathbf{i}}{401} = 3.9900\mathbf{i} \in \langle \lambda_1, \lambda_5 \rangle$ and $|t_{1,2}| = 3.9900 \neq 3.9904 = \lambda_{\max}(H(e^{-\mathbf{i}\theta_1} C_{1,2}))$, i.e., $t_{1,2}$ is interior point of $NR[C_{1,2}]$.

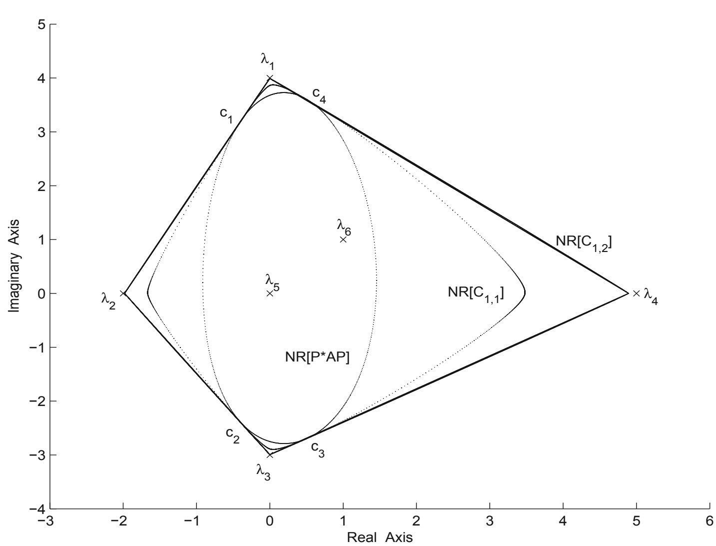
If $\zeta_3 = v + \frac{100}{\sqrt{15}} e_5$, we have $\tilde{R}_{1,3} = \begin{bmatrix} -0.8944 & -0.1826 & -0.3162 & -0.2580 \\ 0.4472 & -0.3651 & -0.6325 & -0.5160 \\ 0 & 0.9129 & -0.3162 & -0.2580 \\ 0 & 0 & 0.6325 & -0.7740 \\ 0 & 0 & 0 & 0.0387 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and

$C_{1,3} = \tilde{R}^*_{1,3} A \tilde{R}_{1,3}$. By (14) we take: $t_{1,3} = \frac{40000\mathbf{i}}{10001} = 3.99960004\mathbf{i} \in \langle \lambda_1, \lambda_5 \rangle$ and $|t_{1,3}| = 3.99960004 \approx \lambda_{\max}(H(e^{-\mathbf{i}\theta_1} C_{1,3})) = 3.9996006$. The point $t_{1,3}$ almost lies on the $\partial NR[C_{1,3}]$, i.e., $t_{1,3} \approx f^*_{1,3} C_{1,3} f_{1,3} = 0.00000036896091 + 3.99960058200810\mathbf{i}$ $\in \partial NR[C_{1,3}]$, where $f_{1,3} = \begin{bmatrix} 0.8945 & 0.1825 & 0.3163 & 0.2580 \end{bmatrix}^T$ is the eigenvector of $H(e^{-\mathbf{i}\theta_1} C_{1,3})$ corresponding to $\lambda_{\max}(H(e^{-\mathbf{i}\theta_1} C_{1,3}))$.

If $\zeta_4 = v + \frac{120}{\sqrt{15}} e_5$, we have $\tilde{R}_{1,4} = \begin{bmatrix} -0.8944 & -0.1826 & -0.3162 & -0.2581 \\ 0.4472 & -0.3651 & -0.6325 & -0.5161 \\ 0 & 0.9129 & -0.3162 & -0.2581 \\ 0 & 0 & 0.6325 & -0.7742 \\ 0 & 0 & 0 & 0.0323 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and

$C_{1,4} = \tilde{R}^*_{1,4} A \tilde{R}_{1,4}$. By (14) we take: $t_{1,4} = \frac{57600\mathbf{i}}{14401} = 3.9997\mathbf{i} \in \langle \lambda_1, \lambda_5 \rangle$ and $|t_{1,4}| = 3.9997222 \approx \lambda_{\max}(H(e^{-\mathbf{i}\theta_1} C_{1,4})) = 3.9997225$. Since $q_3 = \frac{100}{\sqrt{15}} < \frac{120}{\sqrt{15}} = q_4$, then $NR[C_{1,3}] \subseteq NR[C_{1,4}]$ and we expect $t_{1,4}$ to approximate $\partial NR[C_{1,4}]$. In fact, $f_{1,4} = \begin{bmatrix} 0.8945 & 0.1826 & 0.3162 & 0.2581 \end{bmatrix}^T$ is eigenvector of $H(e^{-\mathbf{i}\theta_1} C_{1,4})$ and $t_{1,4} \approx f^*_{1,4} C_{1,4} f_{1,4}$ $= 0.00000017808543 + 3.99972250302240\mathbf{i}$.

In the next figure the numerical ranges of the compressions P^*AP, $C_{1,1}$ and $C_{1,2}$ are illustrated.



Let $Q = \langle \lambda_1, \lambda_2, \ldots, \lambda_\nu \rangle$ be subpolygon of $\mathscr{P}$, with $3 \leqslant \nu < k$ and the sequence of vectors of $\mathbb{C}^n$

$$\eta_\mu = \sum_{i=1}^{\nu} \upsilon_i e_i + \varphi_\mu e_j \, ; \;\; j \in \{k+1, \ldots, n\}$$

and suppose furthermore that the eigenvalue λ_j may not belong to Q. Denoting by $G_{1,\mu} = T^*_{1,\mu} D T_{1,\mu}$ the ν-compression of D, then $NR[G_{1,\mu}]$ is tangent to the polygon $\langle \lambda_1, \lambda_2, \ldots, \lambda_\nu, \lambda_j \rangle$. Thus, when $\lim_{\mu \to \infty} \|\eta_\mu\|_2 = \infty$, by Theorem 4 we conclude the equality

$$\lim_{\mu \to \infty} NR[G_{1,\mu}] = Q.$$

Therefore, the separation of polygon $\mathscr{P} = \bigcup_{\delta=1}^{p} Q_\delta$ into p-subpolygons leads to

$$\bigcup_{\delta=1}^{p} \big(\lim_{\mu \to \infty} NR[G^\delta_{1,\mu}] \big) = NR[A],$$

where $G^\delta_{1,\mu}$ is a compression of associated Q_δ, according to Theorem 4.

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