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The Joint Numerical Range of Bordered and Tridiagonal Matrices

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Dedicated to Professor Peter Lancaster, with special appreciation.

Abstract. Let A_m $(m = 1, \ldots, k)$ be $n \times n$ matrices, the joint numerical range is defined by

$$\mathrm{JNR}[A_1, \ldots, A_k] = \{ (x^* A_1 x, \ldots, x^* A_k x) : x \in \mathbb{C}^n , \|x\| = 1 \}.$$

In this paper, some geometric properties of JNR are presented, when the hermitian A_m are bordered or $(2\mu - 1)$– diagonal matrices and the convexity of JNR is investigated.

1. Introduction

Let $\mathcal{M}_n$ be the algebra of $n \times n$ complex matrices. For the matrices $A_1, \ldots, A_k \in \mathcal{M}_n$, the *joint numerical range*, is the set

$$\mathrm{JNR}[A_1, \ldots, A_k] = \{ (x^* A_1 x, \ldots, x^* A_k x) : x \in \mathbb{C}^n , \|x\| = 1 \}. \quad (1.1)$$

It is also called *k-dimensional field of k matrices* [HJ, p.85] and it will be denoted by

$$\mathrm{JNR}[A_m]_{m=1}^k .$$

Clearly, for $k = 1$ the joint numerical range is identified with the numerical range of the matrix A_1. Also, for any matrix $A \in \mathcal{M}_n$

$$\mathrm{JNR}\left[\frac{A + A^*}{2}, \frac{A - A^*}{2i}\right] = \mathrm{NR}[A].$$

The joint numerical range is not necessarily convex, but the convexity of joint numerical range is known for hermitian matrices when

$$n = k = 2 \quad \textbf{and} \quad n \geq 3,\ k \leq 3.$$

The convex hull of joint numerical range will be denoted by $\mathrm{Co}\{\mathrm{JNR}[A_m]_{m=1}^k\}$.

The size of $\mathrm{JNR}[A_m]_{m=1}^k$ is measured by the smallest sphere centered at the origin that contains it. Clearly, the radius of the sphere, known as the *numerical radius*, is defined by:

$$r_k = sup\{ |z| : z \in \mathrm{Co}\{\mathrm{JNR}[A_m]_{m=1}^k\}\}.$$

Recently a connection of $\mathrm{Co}\{\mathrm{JNR}[A_m]_{m=1}^k\}$ with the numerical range of matrix polynomial

$$P(\lambda) = A_k\lambda^{k-1} + \ldots + A_2\lambda + A_1$$

has been presented in [PT]. Precisely, it is noticed that

$$\begin{aligned} \mathrm{NR}[P(\lambda)] &= \{\lambda \in \mathbf{C} : x^*P(\lambda)x = 0,\ x \in \mathbf{C}^n \setminus \{0\}\} \\ &= \{\lambda \in \mathbf{C} : c_k\lambda^{k-1} + \ldots + c_2\lambda + c_1 = 0, \\ &\qquad (c_1, c_2, \ldots, c_k) \in \mathrm{Co}\{\mathrm{JNR}[A_m]_{m=1}^k\}\}. \end{aligned}$$

In this paper, we investigate geometric properties of joint numerical range of special matrices which occur in graph theory. In the next section the first statement is that the numerical range of $n\times n$ bordered matrix is an elliptic disk. Afterwards, this idea is undertaken for the joint numerical range of a family of hermitian bordered matrices to be hyperellipsoid with nonempty interior.

The last section is devoted to the same problem for 3×3 tridiagonal hermitian matrices and the according results are presented.

2. The NR of bordered matrices

Using the term *"bordered matrix"* we consider the matrices of the form

$$S = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & 0 & 0 & \ldots & 0 \\ a_{31} & 0 & & & \\ \vdots & \vdots & & \mathbf{O} & \\ a_{n1} & 0 & & & \end{bmatrix} \tag{2.1}$$

which correspond to "star" graphs [BR]. Clearly, for $n = 3$ interchanging first and second rows and columns, the resulting matrix PSP^T is the tridiagonal matrix,

$$\begin{bmatrix} 0 & a_{21} & 0 \\ a_{12} & a_{11} & a_{13} \\ 0 & a_{31} & 0 \end{bmatrix}$$

where P is permutation matrix. Generalizing the result in [C] on the numerical range of tridiagonal matrices with zero diagonals, we say:

Lemma 2.1. *The numerical range of matrix*

$$A = \begin{bmatrix} a & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a \end{bmatrix} \tag{2.2}$$

is an elliptical disk, centered at the point $\frac{1}{2}(a + a_{22})$ *of the complex plane.*

Proof. It is enough to consider the matrix

$$A_0 = \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & \hat{a}_{22} & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} \quad ; \quad \hat{a}_{22} = a_{22} - a$$

since $\mathrm{NR}[A] = \mathrm{NR}[A_0] + a$. For the matrix A_0, after algebraic manipulations, we verify that

$\sigma(A_0) = \{0, \dfrac{\hat{a}_{22} + \Delta^{1/2}}{2}, \dfrac{\hat{a}_{22} - \Delta^{1/2}}{2}\}$ with $\Delta = \hat{a}_{22}^2 + 4(a_{21}a_{12} + a_{23}a_{32})$ and

I. $q = tr(A_0^*A_0) - \sum_{j=1}^3 |\lambda_j|^2 \geq (|a_{21}| - |a_{12}|)^2 + (|a_{23}| - |a_{32}|)^2 > 0$

II. $q\, trA_0 + \sum_{j=1}^3 |\lambda_j|^3\lambda_j - tr(A_0^*A_0^2) = 0$

III. $(|\lambda_2| + |\lambda_3|)^2 - |\lambda_2 - \lambda_3|^2 \leq q$

Therefore, by [KRS], the $\mathrm{NR}[A_0]$ is an ellipse with center $\frac{1}{2}\hat{a}_{22}$. □

Following using Lemma 2.1 and a result in [KRS] we say:

Proposition 2.2. *The numerical range of S in (2.1) is an elliptic disk centered at the point $\frac{a_{11}}{2}$ with axes of length $(\alpha \pm |\beta|)^{1/2}$, where*

$$\alpha = \frac{|a_{11}|^2}{2} + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} |a_{ij}|^2 \quad , \quad \beta = \frac{a_{11}^2}{2} + 2\sum_{j=2}^{n} a_{1j}a_{j1}. \tag{2.3}$$

Proof. Denoting by $\hat{\alpha} = [a_{21}\, a_{31} \ldots a_{n1}]^T$ and $Q = I - 2\dfrac{yy^*}{y^*y}$ the Householder matrix, for $y = \hat{\alpha} - k_1e_1$, then

$$Q\hat{\alpha} = k_1e_1 \quad ; \quad k_1 = \frac{\|\hat{\alpha}\|}{|a_{21}|}\, a_{21}$$

and

$$RSR^* = \begin{bmatrix} a_{11} & b_{12} & \ldots & b_{1n} \\ k_1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & \mathbf{O} & \\ 0 & & & \end{bmatrix}$$

where $R = diag(1, Q)$. If $R_1 = diag(1, 1, Q_1)$, with Q_1 the Householder matrix such that

$$Q_1 \begin{bmatrix} \bar{b}_{13} \\ \vdots \\ \bar{b}_{1n} \end{bmatrix} = k_2e_1 \quad ; \quad k_2 = \frac{\|[b_{13}, \ldots, b_{1n}]\|}{|b_{13}|}\,\bar{b}_{13}$$

then

$$(R_1R)S(R_1R)^* = \hat{S} \oplus O_{n-3} \quad ; \quad \hat{S} = \begin{bmatrix} a_{11} & b_{12} & \bar{k}_2 \\ k_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly $0 \in \mathrm{NR}[\hat{S}]$, $\mathrm{NR}[S] = \mathrm{NR}[\hat{S}]$ and by Lemma 2.1, the $\mathrm{NR}[\hat{S}]$ is an ellipse centered at the point $a_{11}/2$.
To calculate the axes of ellipse, by the elements of S in (2.1), consider its focii $\lambda_{1,2} = \dfrac{a_{11} \pm \sqrt{\Delta}}{2}$ with $\Delta = a_{11}^2 + 4k_1b_{12}$ and the quantity

$$\delta = tr(\hat{S}^*\hat{S}) - \sum |\lambda_{\hat{S}}|^2 = |k_1|^2 + |k_2|^2 + |b_{12}|^2 + \frac{|a_{11}|^2}{2} - \frac{|\Delta|}{2}.$$

Clearly,

$$\begin{aligned} \delta &= \frac{|a_{11}|^2}{2} + \sum_{j=2}^{n} |a_{j1}|^2 + \sum_{j=2}^{n} |b_{1j}|^2 - \frac{1}{2}\left| a_{11}^2 + 4[\, b_{12} \ldots b_{1n}\,] k_1 e_1 \right| \\ &= \frac{|a_{11}|^2}{2} + \sum_{j=2}^{n} |a_{j1}|^2 + \|\, [\, a_{12} \ldots a_{1n}\,] Q^* \|^2 - \frac{1}{2}\left| a_{11}^2 + 4[\, a_{12} \ldots a_{1n}\,] Q^*Q\, \hat{\alpha} \right| \\ &= \frac{|a_{11}|^2}{2} + \sum_{j=2}^{n} \left(|a_{j1}|^2 + |a_{1j}|^2 \right) - \frac{1}{2}\left| a_{11}^2 + 4[a_{12} \ldots a_{1n}]\, \hat{\alpha} \right| \\ &= \alpha - |\beta| \end{aligned}$$

and $|\lambda_2 - \lambda_1|^2 = |\Delta| = 2|\beta|$. Therefore, the axes of the ellipse are

$$\delta^{1/2} \quad \text{and} \quad \left(\delta^2 + |\lambda_2 - \lambda_1|^2 \right)^{1/2} = (\alpha + |\beta|)^{1/2}. \qquad \square$$

Since for any $z \in \mathrm{NR}[S]$, $|z - \dfrac{a_{11}}{2}| \leq \dfrac{(\alpha + |\beta|)^{1/2}}{2}$, we obtain:

Corollary 2.3. *The numerical radius of the bordered matrix S in (2.1) is equal to*

$$\frac{|a_{11}| + (\alpha + |\beta|)^{1/2}}{2}.$$

Evidently

$$\hat{S} = \begin{bmatrix} & & & a_{n1} & & & \\ & \mathbf{O} & & \vdots & & \mathbf{O} & \\ & & & a_{n-s+1,1} & & & \\ a_{1n} & \ldots & a_{1,n-s+1} & a_{11} & a_{12} & \ldots & a_{1,n-s} \\ & & & a_{21} & & & \\ & \mathbf{O} & & \vdots & & \mathbf{O} & \\ & & & a_{n-s,1} & & & \end{bmatrix}$$

is similar to S by a permutation matrix and is the sub-direct sum of two bordered matrices S_1, S_2. The ellipses $E_i = \mathrm{NR}[S_i]$ have the point $\dfrac{a_{11}}{2}$ as their common center and their axes are equal to $(\alpha_i \pm |\beta_i|)^{1/2}$, where

$$\alpha_1 = \frac{|a_{11}|^2}{2} + \sum_{\substack{i,j=1 \\ i \neq j}}^{n-s} |a_{ij}|^2 \quad , \quad \beta_1 = \frac{a_{11}^2}{2} + 2\sum_{j=2}^{n-s} a_{1j}a_{j1}$$

$$\alpha_2 = \frac{|a_{11}|^2}{2} + \alpha - \alpha_1 \qquad , \qquad \beta_2 = \frac{a_{11}^2}{2} + \beta - \beta_1.$$

Since

$$\hat{S} = \begin{bmatrix} & \vdots & \mathbf{O} \\ \ldots & a_{11}/2 & \\ \mathbf{O} & & \mathbf{O} \end{bmatrix} + \begin{bmatrix} \mathbf{O} & & \mathbf{O} \\ & a_{11}/2 & \ldots \\ \mathbf{O} & \vdots & \end{bmatrix} = \hat{S}_1 + \hat{S}_2$$

we remark that:

$$\mathrm{Co}\{\mathrm{NR}[S_1] \cup \mathrm{NR}[S_2]\} \subseteq \mathrm{NR}[S] \subseteq \mathrm{NR}[\hat{S}_1] + \mathrm{NR}[\hat{S}_2].$$

3. The JNR of bordered matrices

Given the $n \times n$ hermitian matrices $A_1, A_2, \ldots, A_k$, we consider the matrix

$$M = h_1A_1 + h_2A_2 + \ldots + h_kA_k \tag{3.1}$$

for any unit vector $\mathbf{h} = (h_1, h_2, \ldots, h_k) \in \mathbf{R}^k$.

The approximation of $\mathrm{JNR}[A_m]_{m=1}^k$ by convex polyhedra is owed by the next lemma.

Lemma 3.1. *The support plane*

$$\varepsilon : h_1x_1 + h_2x_2 + \ldots + h_kx_k = \lambda_{max}(M)$$

of the surface $\partial Co\{JNR[A_m]_{m=1}^k\}$ is tangent at the point

$$\rho_0 = (w_0^*A_1w_0,\ w_0^*A_2w_0, \ldots, w_0^*A_kw_0)$$

where w_0 is the corresponding unit eigenvector of $\lambda_{max}(M)$.

This statement is a generalization of the cases $k = 2$ or $k = 3$ in [AT] and it is confirmed in a similar way. If the $\mathrm{JNR}[A_m]_{m=1}^k$ is not convex then there exists a support plane ε having more than one common points with $\mathrm{Co}\{\mathrm{JNR}[A_m]_{m=1}^k\}$ and evidently all curves of $\partial\,\mathrm{Co}\{\mathrm{JNR}[A_m]_{m=1}^k\}$ are not differentiable at any point.

The investigation of convexity of JNR has been treated by Li-Poon and there the problem to characterize and determine the maximal linearly independent convex family, for the JNR of hermitian matrices, has been put forward. It has been proved in [LP, Th.2.3] that, for a linearly independent k-tuple of $n \times n$ hermitian matrices A_m, if the $\mathrm{JNR}[A_m]_{m=1}^k$ is convex then

$$dim(span\{I, A_1, \ldots, A_k\}) \leq 2n - 1.$$

Following we consider a family of *hermitian bordered* matrices

$$S_m = \begin{bmatrix} a_{m1} & a_{m2} & \dots & a_{mn} \\ \bar{a}_{m2} & 0 & \dots & 0 \\ \vdots & \vdots & \mathbf{O} & \\ \bar{a}_{mn} & 0 & & \end{bmatrix} \quad ; \quad m = 1, \dots, k \tag{3.2}$$

for $n \geq 3$ and $3 \leq k \leq 2n-1$. The restriction of k comes from the observations that $S_m \in W = span\{E_{11}, E_{1j}+E_{j1}, i(E_{1j}-E_{j1}) \; : \; 2 \leq j \leq n\}$, where $\{E_{ij}\}$ is the standard basis of $\mathcal{M}_n$ and [LP]

$$\text{JNR}[S_m]_{m=1}^k \text{ is convex } \Leftrightarrow$$

$$\Leftrightarrow \text{JNR}[E_{11}, E_{1j}+E_{j1}, i(E_{1j}-E_{j1}); \; 2 \leq j \leq n] \text{ is convex.}$$

Proposition 3.2. *Let $\{S_m\}_{m=1}^k$ be a linearly independent family of matrices in (3.2). The boundary of $Co\{JNR[S_m]_{m=1}^k\}$ is an hyperellipsoid in $\mathbf{R}^k$ with center $\frac{1}{2}(a_{11}, a_{21}, \dots, a_{k1})$.*

Proof. For any unit vector $(h_1, \dots, h_k) \in \mathbf{R}^k$ we consider as in (3.1), the hermitian matrix

$$M = h_1 S_1 + h_2 S_2 + \dots + h_k S_k.$$

Since

$$|\lambda I - M| = \lambda^n - \lambda^{n-1} \sum_{m=1}^k h_m a_{m1} - \lambda^{n-2} \sum_{j=2}^n \left| \sum_{m=1}^k h_m a_{mj} \right|^2$$

we have

$$\lambda_{max}(M) = \lambda_0 = \frac{1}{2}\left(\sum_{m=1}^k h_m a_{m1} + \Delta^{1/2} \right)$$

where $\Delta = \left(\sum_{m=1}^k h_m a_{m1}\right)^2 + 4\sum_{j=2}^n \left|\sum_{m=1}^k h_m a_{mj}\right|^2 > 0$. The corresponding unit eigenvector of M is

$$w_0 = \lambda_0^{-1/2} \Delta^{-1/4} \begin{bmatrix} \lambda_0 & \sum_{m=1}^k h_m \bar{a}_{m2} & \dots & \sum_{m=1}^k h_m \bar{a}_{mn} \end{bmatrix}^T. \tag{3.3}$$

Hence, by the lemma 3.1, the point $(w_0^* S_1 w_0, \dots, w_0^* S_k w_0)$ belongs to

$$\partial \, \text{Co} \, \{\text{JNR}[S_m]_{m=1}^k\}$$

and the r-coordinate of this point is equal to

$$w_0^* S_r w_0 = \frac{a_{r1}}{2} + \frac{1}{\sqrt{\Delta}} \left[\frac{a_{r1}}{2} \sum_{m=1}^k h_m a_{m1} + \right. \tag{3.4}$$

$$\left. + \sum_{j=2}^n \left[(\sum_{m=1}^k h_m a_{mj}) \bar{a}_{rj} + (\sum_{m=1}^k h_m \bar{a}_{mj}) a_{rj} \right] \right]. \tag{3.5}$$

Setting in (3.5)

$$\pi_r = 2 \left[\frac{a_{r1}^2}{4} + \sum_{j=2}^n |a_{rj}|^2 \right] \Delta^{-1/2}$$

$$\sigma_{r,l} = \sigma_{l,r} = 2 \left[Re(\sum_{j=2}^n a_{rj} \bar{a}_{lj}) + \frac{a_{r1} a_{l1}}{4} \right] \Delta^{-1/2}$$

$$\begin{aligned} h_1 &= cos\theta_1 sin\theta_2 \dots sin\theta_{k-1} \\ h_2 &= sin\theta_1 sin\theta_2 \dots sin\theta_{k-1} \\ h_3 &= cos\theta_2 sin\theta_3 \dots sin\theta_{k-1} \quad ; \quad 0 \leq \theta_i \leq 2\pi, \; i = 1, 2, \dots, k-1 \\ h_4 &= cos\theta_3 sin\theta_4 \dots sin\theta_{k-1} \\ &\vdots \\ h_k &= cos\theta_{k-1}, \end{aligned}$$

by the k equations

$$\begin{aligned} h_1\pi_1 + h_2\sigma_{12} + \dots + h_k\sigma_{1k} &= w_0^* S_1 w_0 - \frac{a_{11}}{2} = x_1 \\ h_1\sigma_{21} + h_2\pi_2 + \dots + h_k\sigma_{2k} &= w_0^* S_2 w_0 - \frac{a_{21}}{2} = x_2 \\ &\vdots \qquad\qquad \vdots \\ h_1\sigma_{k1} + h_2\sigma_{k2} + \dots + h_k\pi_k &= w_0^* S_k w_0 - \frac{a_{k1}}{2} = x_k, \end{aligned} \tag{3.6}$$

due to $\|\mathbf{h}\| = 1$, the parameters $\theta_1, \dots, \theta_{k-1}$ are eliminated and we obtain the quadratic form

$$g_{11}x_1^2 + g_{22}x_2^2 + \dots + g_{kk}x_k^2 + 2\sum_{i=1}^{k-1}\sum_{j=1}^{k-i} g_{i(i+j)} x_i x_{i+j} = g_0^2. \tag{3.7}$$

In (3.7) $g_{mm} = v_m \circ v_m$, $g_{i(i+j)} = v_i \circ v_{i+j}$; $v_m = [\tau_{m1} \; \tau_{m2} \; \dots \; \tau_{mk}]$ where τ_{ms} is the cofactor of ms-element of matrix

$$G = \begin{bmatrix} \pi_1 & \sigma_{12} & \dots & \sigma_{1k} \\ \sigma_{21} & \pi_2 & \dots & \sigma_{2k} \\ \vdots & & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \dots & \pi_k \end{bmatrix}$$

and $g_0 = det\, G$. The matrix

$$Q = \begin{bmatrix} g_{11} & g_{12} & \dots & g_{1k} \\ g_{21} & g_{22} & \dots & g_{2k} \\ \vdots & & \ddots & \vdots \\ g_{k1} & g_{k2} & \dots & g_{kk} \end{bmatrix}$$

of quadratic form (3.7) is a real nonsingular Gram matrix. Since for the system equations in (9) $rankG = k$, we have $rank(adjG) = rank[v_1 \dots v_k] = k$ and hence $Q > 0$. Therefore the form in (3.7) is an hyperellipsoid in $\mathbf{R}^k$ centered at the point $\frac{1}{2}(a_{11}, a_{21}, \dots, a_{k1})$. □

Furthermore, by Proposition 3.2, since every point on the boundary of

$$\mathrm{Co}\{\mathrm{JNR}[S_m]_{m=1}^k\}$$

is an extreme point, it is worth noting:

Corollary 3.3. *The boundary points of $Co\{JNR[S_m]_{m=1}^k\}$ are identified with the shell (outer boundary) of $JNR[S_m]_{m=1}^k$.*

Note that, if $S_1, \dots, S_k$ are not linearly independent matrices and let $S_k = \sum_{i=1}^{k-1} l_i S_i$, then after some algebraic manipulation we have that the last column and row of G is equal to zero and consequently the quadratic form (3.7) degenerates to zero.

Proposition 3.4. *If the origin is not an outer boundary point of $JNR[S_m]_{m=1}^k$, then the interior of $JNR[S_m]_{m=1}^k$ is nonempty.*

Proof. By the bordered matrices $S_1, S_2, \dots, S_k$ we consider the matrix polynomial

$$P(\lambda) = S_k\lambda^{k-1} + \dots + S_2\lambda + S_1.$$

Since any vector $x \in span\{e_2, \dots, e_n\}$ is common isotropic of S_m, i.e. $x^* S_m x = 0$, then $\mathrm{NR}[P(\lambda)] = \mathbf{C}$ and especially $(0, \dots, 0)$ is inside $\mathrm{JNR}[S_m]_{m=1}^k$. Denoting by

$$\mathcal{L}(t) = \{(c_1, c_2, \dots, c_k) \; : \; c_k t^{k-1} + \dots + c_2 t + c_1 = 0, \; (c_1, \dots, c_k) \in \mathrm{JNR}[S_m]_{m=1}^k\}$$

for any $t \in \mathrm{NR}[P(\lambda)]$, $\mathcal{L}(t)$ is a connected set in [PT] and $(0, \dots, 0) \in \mathcal{L}(t)$. Thus, for an outer boundary point $w = (w_1, \dots, w_k)$ of $\mathrm{JNR}[S_m]_{m=1}^k$ we define the polynomial

$$q_w(\lambda) = w_k\lambda^{k-1} + \dots + w_2\lambda + w_1$$

which is annihilating at t_0 and due to connectivity of $\mathcal{L}(t_0)$ there exists a curve $\mathcal{K}$ joining the origin and w. All points of the curve $\mathcal{K}$ belong also to $\mathrm{JNR}[S_m]_{m=1}^k$, since by the continuous function

$$\Phi \; : \; \{x \in \mathbf{C}^n \; : \; \|x\| = 1\} \longrightarrow \mathrm{JNR}[S_m]_{m=1}^k$$

defined by

$$\Phi(x) = (x^* S_1 x, \dots, x^* S_k x)$$

we have

$$\Phi\,[\{x \in \mathbf{C}^n \; : \; \|x\| = 1 \;,\; x^* P(t_0) x = 0\}] = \mathcal{L}(t_0)$$ □

If we don't use the hypothesis for the origin, then the $\mathrm{JNR}[S_m]_{m=1}^k$ might be a shell without interior, as we see in the next example.

Example 3.5. Let the bordered matrices

$$S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

and $x = [\,t\,,\, e^{i\theta}\sqrt{1-t^2}\,]^\top$, $t \in [0,1]$. Then

$$\begin{aligned} \mathrm{JNR}[S_1, S_2, S_3] &= \{(t^2,\, 2t\sqrt{1-t^2}\cos\theta,\, -2t\sqrt{1-t^2}\sin\theta) \; : \\ &\qquad 0 \le t \le 1,\; 0 \le \theta \le 2\pi\} \\ &= \{(x, y, z) \; : \; 4(x-1/2)^2 + y^2 + z^2 = 1\}. \end{aligned}$$

Clearly, $(0, 0, 0) \in \partial\,\mathrm{JNR}[S_1, S_2, S_3]$ and the $\mathrm{JNR}[S_1, S_2, S_3]$ is the surface of ellipsoid.

Following the Propositions 3.2 and 3.4 we conclude that:

Proposition 3.6. *If $S_1, \dots, S_k$ be an k-tuple of linearly independent hermitian bordered matrices with $a_{m1} = 0$ $(m = 1, \dots, k)$, then the $JNR[S_m]_{m=1}^k$ is a convex set.*

Proof. Clearly, $S_m \in V = span\{E_{1j} + E_{j1}, i(E_{1j} - E_{j1}) \; : \; 2 \le j \le n\}$ and $dim(V \cup \{I\}) = 2n-1$. It is enough to see that the points $t\,(w_0^* S_1 w_0, \dots, w_0^* S_k w_0)$, $t \in [0,1]$ belong to $\mathrm{JNR}[S_m]_{m=1}^k$. Setting $\hat{S}_m = tS_m$ and $\hat{M} = h_1\hat{S}_1 + \dots + h_k\hat{S}_k$, we have

$$\lambda_{max}(\hat{M}) = t\,\lambda_{max}(M) = t\lambda_0, \qquad \hat{\Delta} = t^2\Delta$$

and w_0 in (3.3) is the corresponding unit eigenvector of $\lambda_{max}(\hat{M})$. Hence $(w_0^* t S_1 w_0, \dots, w_0^* t S_k w_0)$ is interior point of $\mathrm{JNR}[S_m]_{m=1}^k$. □

Finally, since $dim(V \cup \{I\}) = 2n-1$, we confirm that the matrices $\{S_m\}$ in (3.2) with $a_{m1} = 0$ form a maximal linearly independent family with maximum number of elements, such that the $\mathrm{JNR}[S_m]_{m=1}^k$ is convex.

4. The NR of $(2\mu - 1)$-diagonal matrices

Compared with serial systems, parallel systems permit more freedom of expression in problem analysis and programming. A foundation in the skills of thinking in parallel is basic to the understanding of such systems. Computations within each level are performed in parallel and this simple idea is the basis of recursion, where the total computation is repeatedly divided into separate computations of equal complexity that can be executed in parallel. The natural means of carrying out these operations is to use a μ-tree inter connection of processors. These trees correspond to the $(2\mu-1)$-diagonal matrix $T \in \mathcal{M}_n$, [M]:

$$T = \begin{bmatrix} a & a_{12} & \cdots & a_{1,\mu} & 0 & \\ a_{21} & a_{22} & \cdots & a_{2,\mu} & a_{2,\mu+1} & \mathbf{O} \\ \vdots & \vdots & & & \ddots & \\ a_{\mu 1} & a_{\mu 2} & \cdots & a_{\mu\mu} & & a_{n-\mu+1,n} \\ 0 & a_{\mu+1,2} & & & & \vdots \\ & \ddots & & & a_{n-1,n-1} & a_{n-1,n} \\ \mathbf{O} & & a_{n,n-\mu+1} & \cdots & a_{n,n-1} & a \end{bmatrix}$$

Clearly, for any unit vector $x = x_1e_1 + (x_2e_2 + \ldots + x_{n-1}e_{n-1}) + x_ne_n = x_1e_1 + \xi + x_ne_n = x_1e_1 + \|\xi\|\zeta + x_ne_n = [e_1 \ \ \zeta \ \ e_n]\,[x_1 \ \ \|\xi\| \ \ x_n]^T$, where $\zeta = \dfrac{\xi}{\|\xi\|}$, we obtain

$$x^*Tx = \delta^* \begin{bmatrix} e_1^*Te_1 & e_1^*T\zeta & e_1^*Te_n \\ \zeta^*Te_1 & \zeta^*T\zeta & \zeta^*Te_n \\ e_n^*Te_1 & e_n^*T\zeta & e_n^*Te_n \end{bmatrix} \delta \ ; \ \delta = [x_1 \ \ \|\xi\| \ \ x_n]^T \ , \ \|\delta\| = 1 \qquad (4.1)$$

and consequently

$$\mathrm{NR}[T] = \bigcup_{e_1,\zeta,e_n} \mathrm{NR}\left(\begin{bmatrix} e_1^*Te_1 & e_1^*T\zeta & e_1^*Te_n \\ \zeta^*Te_1 & \zeta^*T\zeta & \zeta^*Te_n \\ e_n^*Te_1 & e_n^*T\zeta & e_n^*Te_n \end{bmatrix} \right).$$

In (4.1), note that,

$$e_1^*Te_1 = e_n^*Te_n = a \ , \quad e_1^*Te_n = e_n^*Te_1 = 0$$

i.e. the matrix is tridiagonal with diagonal entries a, $\zeta^*T\zeta$, a. Hence, by Lemma 2.1 we have:

Corollary 4.1. *The numerical range of $(2\mu-1)$-diagonal matrices is union of ellipses, which arise from 3×3 compression matrix in (4.1).*

The above tridiagonal matrix is more appropriate than the 2×2 matrix for the approximation of $\mathrm{NR}[T]$ and this is illustrated in the following example.

Example 4.2. Let the matrix

$$T = \begin{bmatrix} 3 & 1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 5 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & -3 & 3 & 3 \end{bmatrix}$$

In the next figure, $\mathrm{NR}[T]$ on the left is approximated by 6 ellipses (NR of 3×3 tridiagonal matrices) and on the right the number of ellipses is double (NR of 2×2 matrices). The MATLAB procedure is presented before the references.

Moreover, due to the transformation of a 3×3 bordered matrix to a tridiagonal matrix and by Propositions 3.2 and 3.6, we have:

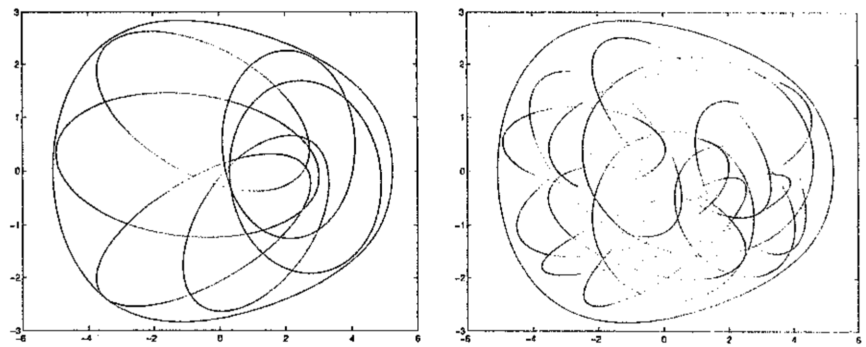

FIGURE 1. *Approximation of* $\mathrm{NR}[T]$.

Proposition 4.3. *For $3 \le k \le 5$, the $\partial\, JNR[T_m]_{m=1}^k$ of the linearly independent hermitian and tridiagonal matrices*

$$T_m = \begin{bmatrix} f_m & a_m & 0 \\ \bar{a}_m & b_m & c_m \\ 0 & \bar{c}_m & f_m \end{bmatrix} , \qquad (4.2)$$

is an hyperellipsoid in $\mathbf{R}^5$. If $f_m = b_m = 0$, then the $JNR[T_m]_{m=1}^k$ is convex set.

The presentation of JNR as a union of joint numerical ranges of matrices of dimension less than n is a convenient way to approach the problem, as noted for the numerical range of matrices in [MA]. In fact:

Lemma 4.4. *For the matrices $A_1, A_2, \ldots, A_k \in \mathcal{M}_n$*

$$JNR[A_m]_{m=1}^k = \bigcup_{\substack{\xi_1,\ldots,\xi_\tau \\ \tau<n}} JNR\left(\begin{bmatrix} \xi_1^*A_m\xi_1 & \cdots & \xi_1^*A_m\xi_\tau \\ \vdots & & \vdots \\ \xi_\tau^*A_m\xi_1 & \cdots & \xi_\tau^*A_m\xi_\tau \end{bmatrix} \right)_{m=1}^k \qquad (4.3)$$

where $\xi_1,\ldots,\xi_\tau$ run over all sets by τ orthonormal vectors of $\mathbf{C}^n$.

Proof. For any unit vector x, the vector $y = P^*x$, where $P = [\xi_1 \ \xi_2 \ \ldots \ \xi_\tau]$, $P^*P = I_\tau$, is also unit and

$$x^*A_mx = y^* \begin{bmatrix} \xi_1^*A_m\xi_1 & \cdots & \xi_1^*A_m\xi_\tau \\ \vdots & & \vdots \\ \xi_\tau^*A_m\xi_1 & \cdots & \xi_\tau^*A_m\xi_\tau \end{bmatrix} y.$$

Therefore, the (4.3) is obvious. □

One easily deduces the following corollary by the Lemma 4.4 and Proposition 4.3:

Corollary 4.5. *For a k-tuple linearly independent $n \times n$ hermitian $(2\mu - 1)$-diagonal matrices the $JNR[T_m]_{m=1}^k$ is union of hyperellipsoids in $\mathbf{R}^5$, where $3 \leq k \leq n^2 - (n-\mu)(n-\mu+1)$.*

If the matrices T_m in (4.2) are real and not symmetric, denoting by

$$Re\text{–JNR}[T_m]_{m=1}^k$$

the joint numerical range of the family $T_1, \dots, T_k$, for real unit vectors $x \in \mathbf{R}^3$, then [HJ, p.85]

$$Re\text{–JNR}[T_m]_{m=1}^k = Re\text{–JNR}\left[\frac{T_m + T_m^T}{2}\right]_{m=1}^k$$

Therefore, by Proposition 4.3 we have:

Corollary 4.6. *The $Re\text{–}JNR[T_m]_{m=1}^k$ for 3×3 linearly independent tridiagonal matrices $T_m = [t_{ij}]$ with $t_{11} = t_{33}$, is an hyperellipsoid in $\mathbf{R}^5$.*

MATLAB procedure:

Step 1. Introduce the $n \times n$ matrix T.

Step 2. Introduce the arbitrary linearly independent vectors $x_1, x_2 \in \mathbf{C}^n$.

Step 3. Orthonormalize this set to ξ_1, ξ_2.

Step 4. Calculate the matrix $A = [\xi_i^* T \xi_j]$, $i, j = 1, 2$.

Step 5. Illustrate the numerical range of matrix A.

Step 6. Repeat this procedure for some other set of vectors.

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