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On the Numerical Range of Rational Matrix Functions

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In this paper we investigate bounds of the numerical range of the derivative of a rational matrix function. Moreover, some results on connectedness are presented.

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1 INTRODUCTION

Let $\mathbb{C}[z]$ ($\mathbb{R}[z]$) be the algebra of polynomials in one variable z with coefficients in $\mathbb{C}(\mathbb{R})$ and let

$$W(z) = \left[\frac{p_{ij}(z)}{q_{ij}(z)}\right]_{i,j=1}^{n} \tag{1}$$

be an $n \times n$ *rational matrix function* (r.m.f.), where the elements $p_{ij}(z), q_{ij}(z) \in \mathbb{C}[z]$ and $q_{ij}(z)$ are not identically zero. In linear system theory, a rational matrix function gives the input–output map and admits a representation

$$W(z) = D + C(zI - A)^{-1}B \tag{2}$$

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if and only if $\deg\{p_{ij}(z)\} \le \deg\{q_{ij}(z)\}, i, j = 1, \ldots, n$ (see for example [3]). For $D = W(\infty), B^T = [0\ 0\ \cdots\ 0\ I],\ C = [H_0\ \cdots\ H_{l-1}]$ and

$$A = \begin{bmatrix} 0 & I & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -A_0 & -A_1 & \ldots & -A_{l-1} \end{bmatrix},$$

the r.m.f. $W(z)$ in (2) is written as

$$W(z) = H(z)L(z)^{-1}, \tag{3}$$

where

$$H(z) = \sum_{j=0}^{l-1} z^j H_j, \qquad L(z) = z^l I + \sum_{j=0}^{l-1} z^j A_j$$

are $n \times n$ *matrix polynomials*.

Denoting by $m(z)$ the least common multiplier of $q_{ij}(z)$ $(i,j = 1, \ldots, n)$, it is clear that

$$W(z) = m(z)^{-1} K(z), \tag{4}$$

where $K(z)$ is a matrix polynomial and $\deg\{m(z)\} \ge \deg\{K(z)\}$. For the remainder of this paper, the degrees of $K(z)$ and $m(z)$ are denoted by n_1 and n_2, respectively. Let $\sigma(m)$ be the set of the roots of $m(z)$. The *numerical range* of the r.m.f. $W(z)$ in (4) is defined by

$$\mathrm{NR}[W(z)] = \{\mu \in \mathbb{C} \setminus \sigma(m) : x^* W(\mu) x = 0, \text{ for some nonzero } x \in \mathbb{C}^n\}.$$

Notice that by (4),

$$\mathrm{NR}[W(z)] = \mathrm{NR}[K(z)] \setminus \sigma(m), \tag{5}$$

and thus $\mathrm{NR}[W(z)]$ is not always closed since in general, $\mathrm{NR}[K(z)] \cap \sigma(m) \ne \emptyset$.

When $W(z) = Iz - A$, $\mathrm{NR}[W(z)]$ coincides with the classical numerical range (or field of values) of the matrix A,

$$\mathrm{NR}[A] = \{x^* A x : x \in \mathbb{C}^n, \quad x^* x = 1\}.$$

Let $\sigma(W) = \{z : \det W(z) = 0\}$ be the *spectrum* of $W(z)$ and let $z_0 \in \sigma(W)$. Then there exists a nonzero vector $x_0 \in \mathbb{C}^n$ such that $W(z_0) \times x_0 = 0$. Hence, $z_0 \in \mathrm{NR}[W(z)]$, i.e.,

$$\sigma(W) \subset \mathrm{NR}[W(z)].$$

In the last few years, the numerical range of matrix polynomials has been studied systematically, and a number of interesting results have been obtained (see e.g. [1,2,6–10]).

In general, the numerical range of a matrix polynomial $K(z)$ is not connected or convex. The distribution of the bounded connected components of $\mathrm{NR}[K(z)]$ plays an important role in the factorization of $K(z)$ (see e.g. [5], [10] and [9]). Bounds for the number of the connected components of $\mathrm{NR}[K(z)]$ are established in [2] and [6], and the location of $\mathrm{NR}[K(z)]$ in a circular annulus centred at the origin is considered in [7].

In Section 2, motivated by the work of M. Marden [4] on scalar polynomials and their derivatives, and the results of [7], we locate the numerical range of the derivative of a r.m.f. $W(z)$ in circular/elliptic annuli. In Section 3, we study the relationship between the numerical range of a matrix polynomial $K(z)$ and the numerical range of its derivative $K'(z)$. Moreover, the connectedness of the numerical range of quadratic matrix polynomials is investigated. We remark that this class of matrix polynomials is one of the most important classes for applications (see [5] and the references therein). Finally, two necessary propositions on scalar polynomials are provided in the Appendix.

2 LOCATION OF NUMERICAL RANGES

Consider the r.m.f. $W(z)$ in (4), with

$$K(z) = K_{n_1} z^{n_1} + \cdots + K_1 z + K_0.$$

Readily one can verify the following properties:

I. $\mathrm{NR}[W(z+\alpha)] = \mathrm{NR}[W(z)] - \alpha$ for any $\alpha \in \mathbb{C}$.
II. $\mathrm{NR}[W(\alpha z)] = \alpha^{-1}\,\mathrm{NR}[W(z)]$ for any nonzero $\alpha \in \mathbb{C}$.
III. If the $m \times n$ matrix S ($m \le n$) has full rank, then

$$\mathrm{NR}[S^* W(z) S] \subseteq \mathrm{NR}[W(z)],$$

and equality holds if $m = n$.
IV. If all the coefficients K_j ($j = 1, \ldots, n_1$) of the matrix polynomial $K(z)$ have a common nonzero isotropic vector $x_0 \in \mathbb{C}^n$, i.e., $x_0^* K_j x_0 = 0$, then

$$\mathrm{NR}[W(z)] = \mathbb{C} \backslash \sigma(m).$$

V. If the r.m.f. $W(z)$ in (1) is real (i.e., $p_{ij}(z), q_{ij}(z) \in \mathbb{R}[z]$), then $\mathrm{NR}[W(z)]$ is symmetric with respect to the real axis.
VI. $\mathrm{NR}[W(z)]$ is bounded if and only if $0 \notin \mathrm{NR}[K_{n_1}]$.
VII. $\mathrm{NR}[W(z)^{-1}] = \mathrm{NR}[W(z)] \backslash \sigma(W)$.

The expression of $W(z)$ in (4) yields the properties **I**–**V** through the matrix polynomial $K(z)$, and for **VI**, it is clear that $\mathrm{NR}[W(z)]$ is bounded if and only if $\mathrm{NR}[K(z)]$ is bounded. A necessary and sufficient condition for the boundedness of $\mathrm{NR}[K(z)]$ is that $0 \notin \mathrm{NR}[K_{n_1}]$ (see [2]). For **VII** (see also Theorem 2.2 in [6]), observe that $z_0 \in \mathrm{NR}[W(z)] \backslash \sigma(W)$ if and only if $0 \in \mathrm{NR}[W(z_0)]$. Therefore by

$$x^* W(z_0)[W(z_0)^*]^{-1} W(z_0)^* x = 0$$

we obtain $(W(z_0)^*x)^* W(z_0)^{-1}(W(z_0)^*x) = 0$, which implies $0 \in \mathrm{NR}[W(z_0)^{-1}]$ and $z_0 \in \mathrm{NR}[W(z)^{-1}]$. Conversely, if $z_0 \in \mathrm{NR}[W(z)^{-1}]$, then $z_0 \notin \sigma(W)$ and $0 \in \mathrm{NR}[W(z_0)^{-1}]$. Thus, $0 \in \mathrm{NR}[W(z_0)]$, [7] and $z_0 \in \mathrm{NR}[W(z)]\backslash\sigma(W)$.

Example 1 Let

$$W(z) = \begin{bmatrix} z/(z-1) & 0 \\ 1/z & 1/(z+1) \end{bmatrix} = \begin{bmatrix} z^2(z+1) & 0 \\ z^2-1 & z^2-z \end{bmatrix} \frac{1}{z(z^2-1)}$$
$$= \frac{1}{z(z^2-1)}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} z^3 + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} z^2 + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\right).$$

Since $0 \in \mathrm{NR}\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right)$, $\mathrm{NR}[W(z)]$ is unbounded (Fig. 1). For

$$W_h(z) = \begin{bmatrix} (1+h)z^3 + z^2 & 0 \\ z^2 - 1 & hz^3 + z^2 - z \end{bmatrix}, \quad h = 0.1, \ 0.2$$

the origin does not belong to the numerical range of the leading coefficient $\left[\begin{smallmatrix} 1+h & 0 \\ 0 & h \end{smallmatrix}\right]$, and thus $\mathrm{NR}[W_h(z)]$ is bounded (Fig. 2).

Denote by $\Delta(c : r, R)$ the circular annulus centred at the point c, with inner radius r and outer radius R, and recall the set of the roots of $m(z)$ of degree n_2, $\sigma(m)$. A proposition on the location of $\mathrm{NR}[W'(z)]$ is the following.

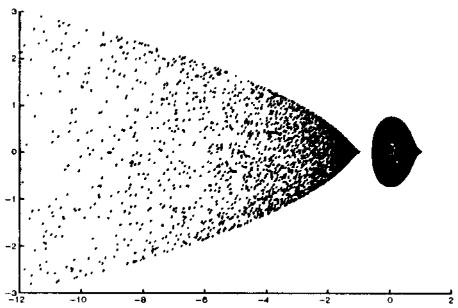

FIGURE 1 The unbounded $\mathrm{NR}[W(z)]$.

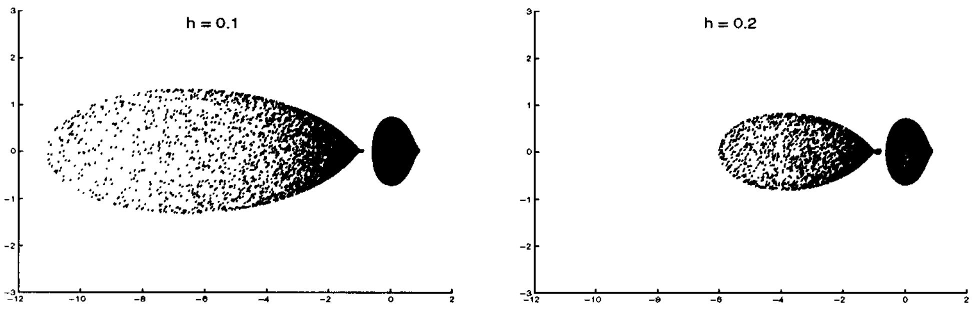

FIGURE 2 Two bounded numerical ranges.

Proposition 1 *Suppose that* $\mathrm{NR}[W(z)]$ *and* $\sigma(m)$ *lie inside two circular annuli,* $\Delta(0 : r_0, R_0)$ *and* $\Delta(0 : r_1, R_1)$*, respectively. Then* $\mathrm{NR}[W'(z)]$ *lies in the circular annulus*

$$D_1 = \Delta\left(0 : \min\{ r_1 , r_0 - R_1 \} ,\ \frac{n_2 R_0 + n_1 R_1}{n_2 - n_1}\right)$$

when $r_0 > R_1$*, or in the circular annulus*

$$D_2 = \Delta\left(0 : \min\{ r_0 , r_1 - R_0 \}, \max\left\{R_1, \frac{n_2 R_0 + n_1 R_1}{n_2 - n_1}\right\}\right)$$

when $R_0 < r_1$.

Proof By (4), it follows that for every nonzero $x \in \mathbb{C}^n$,

$$x^* W'(z)x = \frac{x^* K'(z)x\, m(z) - x^* K(z)x\, m'(z)}{m^2(z)}. \tag{6}$$

Since all the roots of $f_x(z) = x^* K(z)x$ belong to the region $r_0 \le |z| < R_0$, by Proposition I in the Appendix, there exist $\alpha_1(x) \in \Delta(0 : r_0, R_0)$, and $\alpha_2 \in \Delta(0 : r_1 , R_1)$, (notice that $m(z)$ does not depend on x), such that every root μ of the polynomial $x^* W'(z)x$ is equal to $\alpha_1(x)$, or α_2, or

$$\mu = \frac{n_2 \alpha_1(x) - n_1 \alpha_2}{n_2 - n_1}. \tag{7}$$

In the latter case,

$$|\mu| \le \frac{n_2 \left|\alpha_1(x)\right| + n_1 |\alpha_2|}{n_2 - n_1} \le \frac{n_2 R_0 + n_1 R_1}{n_2 - n_1}.$$

Next we consider two cases.
Case **(i)**, suppose $r_0 > R_1$. Then we have $n_2|\alpha_1(x)| \ge n_2 r_0 > n_1 r_0 > n_1 R_1 \ge n_1 |\alpha_2|$, and (7) implies

$$\begin{aligned} |\mu| &\ge \frac{\left| n_2|\alpha_1(x)| - n_1|\alpha_2| \right|}{n_2 - n_1} = \frac{n_2|\alpha_1(x)| - n_1|\alpha_2|}{n_2 - n_1} \\ &\ge \frac{n_2 r_0 - n_1 R_1}{n_2 - n_1} \ge \frac{n_2 r_0 - n_1 R_1}{n_2} > r_0 - R_1. \end{aligned}$$

If $\mu = \alpha_2$, then $|\mu| \ge r_1$ and consequently $|\mu| > \min\{ r_1, r_0 - R_1 \}$. Similarly, if $\mu = \alpha_1(x)$, then $|\mu| < R_0$ and since $R_0 < (n_2 R_0 + n_1 R_1)/(n_2 - n_1)$, clearly $\mu \in D_1$.

Case **(ii)**, suppose $R_0 < r_1$. Then by (7), we obtain

$$\begin{aligned} |\mu| &= \frac{|n_2\alpha_1(x) - n_1\alpha_2|}{n_2 - n_1} > \frac{|n_2\alpha_1(x) - n_1\alpha_2|}{n_2} \\ &= \left|\alpha_1(x) - \frac{n_1}{n_2}\alpha_2\right| > |\alpha_1(x) - \alpha_2| \\ &\ge \left| |\alpha_1(x)| - |\alpha_2| \right| = |\alpha_2| - |\alpha_1(x)| > r_1 - R_0. \end{aligned}$$

Since either $\mu = \alpha_1(x) \in \Delta(0 : r_0, R_0)$, or $\mu = \alpha_2 \in \Delta(0 : r_1, R_1)$, or $r_1 - R_0 < |\mu| \le (n_2 R_0 + n_1 R_1)/(n_2 - n_1)$, we conclude that $\mu \in D_2$. ■

COROLLARY 1 *If* $\Delta(0 : r_0, R_0)$ *and* $\Delta(0 : r_1, R_1)$ *in the above proposition have nonempty intersection, then* $\mathrm{NR}[W'(z)]$ *lies in the disc*

$$S\left(0,\ \max\left\{R_1,\ \frac{n_2 R_0 + n_1 R_1}{n_2 - n_1}\right\}\right).$$

Remark 1 If $\mathrm{NR}[W(z)]$ is bounded, then by [7], we can choose $\Delta(0 : r_0, R_0)$ with

$$r_0 = \frac{\min\limits_{\|x\|=1} |x^* K_0 x|}{\max\limits_{\|x\|=1} |x^* K_0 x| \ +\ \max\limits_{\tau \ne 0}\{\max\limits_{\|x\|=1} |x^* K_\tau x|\}},$$

$$R_0 = 1 + \max_{\tau = 0, 1, \ldots, n_1 - 1}\left\{\max_{\|x\|=1}\left|\frac{x^* K_\tau x}{x^* K_{n_1} x}\right|\right\}.$$

Furthermore, for $m(z)$, we can always consider the circular annulus $\Delta(0 : r_1, R_1)$ in (A.1).

Example 2 Let $W(z)$ be a r.m.f. as in (4), with

$$\begin{aligned} K(z) &= Iz^3 + K_2 z^2 + K_1 z + K_0 \\ &= Iz^3 + \begin{bmatrix} -5 & 2 \\ 0 & -5 \end{bmatrix} z^2 + \begin{bmatrix} 4 & 0 \\ 6 & 4 \end{bmatrix} z + \begin{bmatrix} 3 & 0 \\ 4 & 6 \end{bmatrix} \end{aligned}$$

and $m(z) = (z+1)(z-2)(z^2+5)$. One can verify that $\mathrm{NR}[K_2] = S(-5, 1)$, $\mathrm{NR}[K_1] = S(4, 3)$, and $\mathrm{NR}[K_0] = \{u + iv : u, v \in \mathbb{R}, (u - 4.5)^2/(2.5^2) + v^2/2^2 = 1\}$, and thus,

$$r_0 = \frac{\min\limits_{\|x\|=1} |x^* K_0 x|}{\max\limits_{\|x\|=1} |x^* K_0 x| + \max\limits_{\tau = 1, 2, 3} |x^* K_\tau x|} = \frac{1}{7}$$

and $R_0 = 1 + \max\{7, 7, 6\} = 8$, i.e., $\mathrm{NR}[K(z)] \subset \Delta(0 : \frac{1}{7}, 8)$. For the scalar polynomial $m(z) = z^4 - z^3 + 3z^2 - 5z - 10$, (A.1) implies $r_1 = \frac{2}{3}$ and $R_1 = 11$. Since

$$\begin{aligned} f_x'(z)\, m(z) - f_x(z)\, m'(z) = x^* \Bigg\{ &-z^6 I + \begin{bmatrix} 10 & -4 \\ 0 & 10 \end{bmatrix} z^5 + \begin{bmatrix} -20 & 2 \\ -18 & -20 \end{bmatrix} z^4 + \begin{bmatrix} -14 & 0 \\ -4 & -26 \end{bmatrix} z^3 \\ &+ \begin{bmatrix} -8 & -10 \\ -6 & 1 \end{bmatrix} z^2 + \begin{bmatrix} 82 & -40 \\ -24 & 64 \end{bmatrix} z + \begin{bmatrix} -25 & 0 \\ -40 & -10 \end{bmatrix} \Bigg\} x \end{aligned}$$

and $0 \in \mathrm{NR}\left(\left[\begin{smallmatrix} -25 & 0 \\ -40 & -10 \end{smallmatrix}\right]\right)$, it follows that $0 \in \mathrm{NR}[W'(z)]$ and $\mathrm{NR}[W'(z)] \subset S(0, 55)$.

PROPOSITION 2 *Suppose that* $\mathrm{NR}[W(z)]$ *and* $\sigma(m)$ *lie inside two circular annuli* $\Delta(0 : r_0, R_0)$ *and* $\Delta(c : r_1, R_1)$, *respectively. If* $|c| + R_1 < r_0$, *then* $\mathrm{NR}[W'(z)]$ *lies outside of the ellipse with foci at the origin and c and with major axis* $r_0 - R_1$, *and inside the circle* $S(0, (n_2R_0 + n_1R_1 + n_1|c|)/(n_2 - n_1))$.

Proof For every $\mu \in \mathrm{NR}[W'(z)]$, as in Proposition 1, there are $\alpha_1(x) \in \Delta(0 : r_0, R_0)$ and $\alpha_2 \in \Delta(c : r_1, R_1)$ such that μ coincides with one of these numbers, or by (7), $\mu = (n_2\alpha_1(x) - n_1\alpha_2)/(n_2 - n_1)$.

Then

$$|\mu| \leq \frac{n_2|\alpha_1(x)| + n_1|\alpha_2 - c| + n_1|c|}{n_2 - n_1} < \frac{n_2R_0 + n_1R_1 + n_1|c|}{n_2 - n_1}.$$

If $\mu = \alpha_1(x)$ (or $\mu = \alpha_2$), clearly $|\mu| < R_0$ or $|\mu - c| < R_0$. Otherwise, by (7), we obtain

$$1 < \frac{n_2}{n_1} = \frac{|\mu - \alpha_2|}{|\mu - \alpha_1(x)|} \leq \frac{|\mu - c| + |\alpha_2 - c|}{|\mu - \alpha_1(x)|} \leq \frac{|\mu - c| + R_1}{\big||\mu| - |\alpha_1(x)|\big|}. \tag{8}$$

Moreover, we have $|\mu| < |\alpha_1(x)| < R_0$, because $|\mu| > |\alpha_1(x)| \geq r_0$ implies

$$1 < \frac{|\mu - c| + R_1}{|\mu| - |\alpha_1(x)|} < \frac{|\mu - c| + R_1}{|\mu| - R_0} < 0,$$

a contradiction. Thus, by (8),

$$1 \leq \frac{|\mu - c| + R_1}{|\alpha_1(x)| - |\mu|}$$

and consequently

$$|\mu - c| + |\mu| \geq |\alpha_1(x)| - R_1 \geq r_0 - R_1 > 0.$$

This curve is the prescribed ellipse, since $|c|/(r_0 - R_1) < 1$. ■

Notice that for $c = 0$, the above proposition implies

$$(r_0 - R_1)/2 \leq |\mu| \leq (n_2R_0 + n_1R_1)/(n_2 - n_1).$$

Remark 2 Let the matrix polynomials $K_1(z)$ and $K_2(z)$ of degree n correspond to the r.m.f. $W_1(z)$ and $W_2(z)$, respectively, and

$$\mathrm{NR}[K_j(z)] \subset \Delta(0 : r_j, R_j) \quad (j = 1, 2).$$

Then by Proposition II and the remarks in the Appendix, for every $\mu \in \mathrm{NR}[\lambda_1 K_1(z)+\lambda_2 K_2(z)]$, it follows that

$$\frac{r_1-\omega R_2}{1+\omega} \le |\mu| \le \frac{R_1+\omega R_2}{|1-\omega_k|} \quad \text{when} \quad r_1 > \omega R_2,$$
$$\frac{\omega r_2-R_1}{1+\omega} \le |\mu| \le \frac{R_1+\omega R_2}{|1-\omega_k|} \quad \text{when} \quad R_1 < \omega r_2. \tag{9}$$

Especially, for $\omega = 1$, we have

$$\frac{r_1-R_2}{2} \le |\mu| \le \frac{R_1+R_2}{|1-\omega_k|} \quad \text{when} \quad r_1 > R_2,$$
$$\frac{r_2-R_1}{2} \le |\mu| \le \frac{R_1+R_2}{|1-\omega_k|} \quad \text{when} \quad r_2 \ge R_1. \tag{10}$$

Clearly, (9) and (10) yield *a localisation of the spectrum of* $\lambda_1 K_1(z)+\lambda_2 K_2(z)$ *and hence of the* r.m.f. $\lambda_1 W_1(z)+\lambda_2 W_2(z)$ *in a circular annulus.*

3 A RELATIONSHIP BETWEEN THE NUMERICAL RANGES OF $K(z)$ AND $K'(z)$

Let $W(z)$ be an $n\times n$ r.m.f. as in (4), where $K(z)=K_{n_1}z^{n_1}+\cdots+K_1 z+K_0$. As it is clear from Eq. (5), the investigation of the numerical range of $K(z)$ is substantial for that of $\mathrm{NR}[W(z)]$. Since the poles of $W(z)$ are excluded in the definition of $\mathrm{NR}[W(z)]$, $\mathrm{NR}[W(z)]$ might have more connected components than $\mathrm{NR}[K(z)]$ (when poles of $W(z)$ are node points of the boundary $\partial\,\mathrm{NR}[K(z)]$). Following, we present some propositions relating $\mathrm{NR}[K(z)]$ with $\mathrm{NR}[K(z)]$, which will help us to understand the connectedness of the numerical range of $W(z)$.

PROPOSITION 3 *If* $\mathrm{NR}[K(z)]$ *is bounded and* $\mathrm{NR}[K(z)]\cap\mathrm{NR}[K'(z)]=\emptyset$, *then* $\mathrm{NR}[K(z)]$ *has exactly* n_1 *connected components.*

Proof Since $\mathrm{NR}[K(z)]\cap\mathrm{NR}[K'(z)]=\emptyset$, for every nonzero vector $x\in\mathbb{C}^n$, the polynomial $p_x(z)=x^*K(z)x$ has n_1 disjoint roots. Hence, by [1], $\mathrm{NR}[K(z)]$ has exactly n_1 connected components. ■

Notice that if $\mathrm{NR}[K(z)]$ is bounded and has $k<n_1$ connected components, then

$$\mathrm{NR}[K(z)]\cap\mathrm{NR}[K'(z)] \neq \emptyset.$$

Let now $\lambda_i(x)$ $(i=1,2,\ldots,n_1)$ be the roots of the polynomial

$$p_x(z)=x^*K(z)x,$$

and let Λ_i be their ranges of values. Then $\bigcup_{i=1}^{n_1}\Lambda_i=\mathrm{NR}[K(z)]$, and by [1],

$$\Lambda_i\cap\Lambda_j\cap\mathrm{NR}[K'(z)]\neq\emptyset \quad (i\neq j).$$

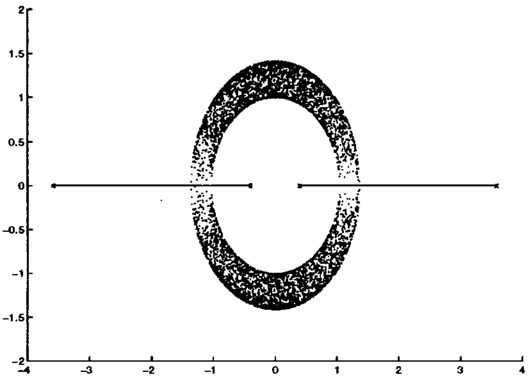

FIGURE 3 A connected numerical range.

It is also worth noting that

$$\bigcup_{1 \le i \le j \le n_1} \left(\Lambda_i \cap \Lambda_j\right) \neq \mathrm{NR}[K(z)] \cap \mathrm{NR}[K'(z)] \tag{11}$$

since a common point ζ of the ranges $\mathrm{NR}[K(z)]$ and $\mathrm{NR}[K'(z)]$ may correspond to different vectors, i.e., $x^*K(\zeta)x = y^*K'(\zeta)y = 0$ for some unit vectors $x \neq y$.

For example, if we consider the matrix polynomial

$$K(z) = Iz^2 + K_1 z + K_0 = Iz^2 + \begin{bmatrix} 0 & 4i \\ -4i & 0 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

then $\mathrm{NR}[K(z)]$ is sketched in Fig. 3, and $\mathrm{NR}[K'(z)] = [-2, 2]$. The ranges of values of $p_x(z)$ are

$$\Lambda_1 = \left\{ \frac{-x^*K_1x + [(x^*K_1x)^2 - 4(x^*K_2x)]^{1/2}}{2} : x \in \mathbb{C}^2,\ x^*x = 1 \right\}$$

and

$$\Lambda_2 = \left\{ \frac{-x^*K_1x - [(x^*K_1x)^2 - 4(x^*K_2x)]^{1/2}}{2} : x \in \mathbb{C}^2,\ x^*x = 1 \right\}.$$

One can see that $2 \in \mathrm{NR}[K(z)] \cap \mathrm{NR}[K'(z)]$ and $2 \in \Lambda_1$, but there is no unit $x_0 \in \mathbb{C}^2$ so that 2 is a double root of $x_0^*K(z)x_0$ and $2 \notin \Lambda_2$. Hence, (11) is verified. Furthermore, by [8], we know that

$$\Lambda_1 \cap \Lambda_2 \cap \mathrm{NR}[K'(z)] \neq \emptyset.$$

Proposition 4 *If the matrix polynomial* $K(z)$ *is selfadjoint (i.e., with Hermitian coefficients) and* $\mathrm{NR}[K(z)] \cap \mathbb{R} = \emptyset$, *then for an arbitrary matrix* B,

$$\mathrm{NR}[L(z)] \cap \mathbb{R} = \emptyset,$$

where

$$L(z) = \frac{1}{n_1+1} K_{n_1} z^{n_1+1} + \cdots + \frac{1}{2} K_1 z^2 + K_0 z + B.$$

Proof Since $L'(z) = K(z)$ and $\mathrm{NR}[K(z)] \cap \mathbb{R} = \emptyset$, there does not exist a nonzero vector $x \in \mathbb{C}^n$ such that the polynomial $x^* L(z) x$ has a double real root. Therefore, by [8], it follows that either $\mathrm{NR}[L(z)] \cap \mathbb{R} = \emptyset$, or $\mathrm{NR}[L(z)] \subset \mathbb{R}$. If $\mathrm{NR}[L(z)] \subset \mathbb{R}$, then by [4, Theorem 6.2], $\mathrm{NR}[K(z)] \subset \mathbb{R}$. This is a contradiction and the proof is complete. ■

Especially, for a self adjoint $K(z) = K_1 z + K_0$, a full description of $\mathrm{NR}[K(z)]$ in terms of the algebraic properties of the coefficients is presented in [2, Theorem 4.1]. This enables us to check if the numerical range of $L(z) = \frac{1}{2} K_1 z^2 + K_0 z + B$ is connected.

Quadratic matrix polynomials of the form $K(z) = Iz^2 + K_1 z + K_0$ arise in many applications, and thus, they are of special interest. Consider the matrix polynomial $M(z)$

$$M(z) = -K(iz) = P(z) - iQ(z), \tag{12}$$

where

$$P(z) = Iz^2 + S(K_1)z - H(K_0), \qquad Q(z) = H(K_1)z + S(K_0)$$

and $H(K) = (K + K^*)/2$, $S(K) = (K - K^*)/(2i)$ are the Hermitian and skew-Hermitian part of matrix K, respectively. Obviously, by (12), we have

$$\mathrm{NR}[M(z)] = \mathrm{NR}[K(iz)] = -i\,\mathrm{NR}[K(z)],$$

i.e.,

$$\mathrm{NR}[M(z)] = e^{-\pi/2i}\ \mathrm{NR}[K(z)]. \tag{13}$$

If

$$\text{conv.hull}\{\,\mathrm{NR}[P(z)] \cap \mathbb{R}\,\} \cap \mathrm{NR}[Q(z)] = \emptyset,$$

then $\mathrm{NR}[M(z)] \cap \mathbb{R} = \emptyset$ and by (13),

$$\mathrm{NR}[K(z)] \cap i\mathbb{R} = \emptyset.$$

Thus, if $\mathrm{NR}[K(z)]$ lies either in the left open half plane, $\mathbb{C}_\ell$, or in the right open half plane, $\mathbb{C}_r$ of $\mathbb{C}$, then $P(z)$ and $Q(z)$ are *hyperbolic* matrix polynomials, i.e., for every $x \in \mathbb{C}^n \backslash \{0\}$,

$$(\,x^* S(K_1) x\,)^2 + 4\,x^* H(K_0) x \;>\; 0$$

and the matrix $H(K_1)$ is definite.

Denoting by $|\lambda(*)|_{\max}$ the maximum absolute value of the eigenvalues of a matrix, then by [7], NR[$P(z)$] is a subset of the circular annulus $\Delta(0:\rho_1,\rho_2)$, where

$$\rho_1 = \frac{\min_{\|x\|=1} |x^* H(K_0)x|}{|\lambda(H(K_0))|_{\max} + \max\{1, |\lambda(S(K_1))|_{\max}\}},$$

and

$$\rho_2 = 1 + \max\{|\lambda(S(K_1))|_{\max}, \ |\lambda(H(K_0))|_{\max}\}.$$

Moreover,

$$\mathrm{NR}[P(z)] \cap \mathbb{R} \subseteq [-\rho_2, \ -\rho_1] \cup [\rho_1, \rho_2],$$

and combining this relationship with the results of [6] and Theorem 4.1 in [2], we have the following.

Remark 3 Consider a quadratic matrix polynomial $K(z)$ as in (12).

I. Suppose that $H(K_1)$ is definite and for every unit vector $x \in \mathbb{C}^n$,

$$\frac{|x^* S(K_0)x|}{|x^* H(K_1)x|} < \rho_1.$$

Then it is easy to see that

$$\{\mathrm{NR}[P(z)] \cap \mathbb{R}\} \cap \mathrm{NR}[Q(z)] = \emptyset.$$

Moreover, by Theorem 1.1 in [6], NR[$P(z)$] is a subset of $\mathbb{C}_r$, or $\mathbb{C}_\ell$, if $P(z)$ is a hyperbolic matrix polynomial. Otherwise, NR[$K(z)$] consists of two connected components.

II. If $H(K_1)$ is definite and for every unit vector $x \in \mathbb{C}^n$,

$$\frac{|x^* S(K_0)x|}{|x^* H(K_1)x|} > \rho_2,$$

then by Theorem 1.2 in [6], NR[$K(z)$] has two connected components, one in the right and one in the left open half plane.

III. If $H(K_1)$ is indefinite, $S(K_0)$ is definite and for the maximum negative eigenvalue λ and minimum positive eigenvalue ξ of the selfadjoint pencil $Q(z)$ in (12), we have

$$\min\{|\lambda|, \xi\} > \rho_2,$$

then by Theorem 1.2 in [6], NR[$K(z)$] consists of two components, one in the right and one in the left open half plane.

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References

[1] Y. Lyubich (1997). Separation of roots of operator polynomials in Hilbert space. *Integral Equations and Operator Theory*, **29**, 52–62.
[2] C.-K. Li and L. Rodman (1994). Numerical range of matrix polynomials. *SIAM J. Matrix Anal. Applic.*, **15**, 1256–1265.
[3] P. Lancaster and L. Rodman (1995). *Algebraic Riccati Equations*. Clarendon Press, Oxford.
[4] M. Marden (1966). Geometry of Polynomials. *Amer. Math. Soc.*, Providence.
[5] A.S. Markus (1988). *Introduction to the Spectral Theory of Polynomial Operator Pencils*, AMS Transl. Math. Monographs, Vol. 71, Amer. Math. Soc., Providence.
[6] J. Maroulas and P. Psarrakos (1998). On the connectedness of numerical ranges of matrix polynomials. *Linear Algebra Appl.*, **280**, 97–108.
[7] J. Maroulas and P. Psarrakos (1997). The boundary of numerical range of matrix polynomials. *Linear Algebra Appl.*, **267**, 101–111.
[8] J. Maroulas and P. Psarrakos (1998). A connection between numerical ranges of selfadjoint matrix polynomials. *Linear and Multilinear Algebra*, **44**, 327–340.
[9] J. Maroulas and P. Psarrakos (2000). On factorization of matrix polynomials. *Linear Algebra Appl.*, **304**, 131–139.
[10] A. Markus and L. Rodman (1998). Some results on numerical ranges and factorization of matrix polynomials. *Linear and Multilinear Algebra*, **42**, 169–185.

APPENDIX

Consider the nth degree scalar polynomial, $f(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n$. As it is known, see [4, pp. 123, 126], all the roots of $f(z)$ lie in the circular annulus

$$D = \{z \in \mathbb{C} : r_1 \leq |z| < R_1\},$$

where

$$r_1 = \min_{\tau=1,\dots,n} \frac{|\alpha_0|}{|\alpha_0| + |\alpha_\tau|} \quad \text{and} \quad R_1 = 1 + \max_{\tau=0,1,\dots,n-1} \left|\frac{\alpha_\tau}{\alpha_n}\right|. \tag{A.1}$$

According to [4], we define as a *circular region* every region in the complex plane, which consists of a closed interior or exterior of a circle.

Proposition I *If all the roots of the polynomial $f_1(z)$ of degree n_1 and the polynomial $f_2(z)$ of degree n_2 belong to the circular regions C_1 and C_2, respectively, then every root z_0 of the polynomial*

$$g(z) = f_1'(z)f_2(z) - f_1(z)f_2'(z) \tag{A.2}$$

is equal to

$$z_0 = \frac{n_2\alpha_1 - n_1\alpha_2}{n_2 - n_1} \tag{A.3}$$

for suitable $\alpha_1 \in C_1$, *and* $\alpha_2 \in C_2$, *or* $z_0 = \alpha_1$, *or* $z_0 = \alpha_2$.

Proof Since $g(z)$ is linear and symmetric in the roots of $f_1(z), f_2(z)$, by Walsh's Theorem [4, p. 62], there exist suitable points $\alpha_1 \in C_1$ and $\alpha_2 \in C_2$, such that every root z_0 of $g(z)$ in (A.2) annihilates the polynomial

$$n_1(z-\alpha_1)^{n_1-1}(z-\alpha_2)^{n_2} - n_2(z-\alpha_1)^{n_1}(z-\alpha_2)^{n_2-1}.$$

If $z_0 \neq \alpha_1, \alpha_2$, then (A.3) is obtained straightforward. ■

PROPOSITION II *Let the polynomials $f_j(z)$ of degree n ($j = 1, 2$) be such that*

$$\sigma(f_j) \subset D_j = \{z \in \mathbb{C} : r_j \leq |z - c_j| \leq R_j\}, \quad j = 1, 2.$$

Then the roots of the linear combination

$$f(z) = \lambda_1 f_1(z) + \lambda_2 f_2(z), \quad \lambda_1 \neq \lambda_2 \neq 0,$$

lie in the sector

$$\left\{z : 0 < |c_0| - \rho \leq |z| \leq |c_0| + \rho; |\mathrm{Arg} z - \mathrm{Arg} c_0| \leq \mathrm{arc\ sin}\left(\frac{\rho}{|c_0|}\right)\right\},$$

where

$$c_0 = \frac{c_1 - \omega^2 c_2}{1 - \omega^2}, \quad \rho = \frac{\omega|c_1 - c_2|}{1 - \omega^2} + \frac{R_1 + \omega R_2}{1 - \omega}$$
$$\text{and} \quad \omega = \left|\sqrt[n]{-\frac{\lambda_2}{\lambda_1}}\right| < 1.^1$$

If $|c_0| \leq \rho$, then the roots of $f(z)$ lie in the disc $|z - c_0| \leq \rho$.

Proof By Theorem 15.4 in [4], (in a similar way as in Theorem 17.1 in [4]), we see that every root μ of $f(z)$ lies in the locus Γ of the roots of the equation

$$\lambda_1(z-\alpha_1)^n + \lambda_2(z-\alpha_2)^n = 0, \tag{A.4}$$

where α_j ($j = 1, 2$) vary independently over D_j ($j = 1, 2$). Thus, by (A.4) it follows

$$\mu = \frac{\alpha_1 - \omega_k \alpha_2}{1 - \omega_k} \quad \left(= \frac{\alpha_2 - \alpha_1 \theta_k}{1 - \theta_k}\right)$$

[1]If $\omega > 1$, then we consider $\theta = |\sqrt[n]{-(\lambda_1/\lambda_2)}| < 1$.

with $\omega_k = (-\lambda_2/\lambda_1)^{1/n},\ (\theta_k = \omega_k^{-1}),\ k = 1, 2, \ldots, n$. Denoting by

$$d_k = \frac{c_1 - \omega_k c_2}{1 - \omega_k},$$

for $|\lambda_2/\lambda_1| < 1,$ we have

$$|\mu - d_k| = \left|\frac{\alpha_1 - \omega_k \alpha_2}{1 - \omega_k} - \frac{c_1 - \omega_k c_2}{1 - \omega_k}\right| \le \frac{|\alpha_1 - c_1| + |\omega_k||\alpha_2 - c_2|}{|1 - \omega_k|}$$
$$\le \frac{R_1 + \omega R_2}{|1 - \omega_k|} \le \frac{R_1 + \omega R_2}{|1 - |\omega_k||} = \frac{R_1 + \omega R_2}{1 - \omega}$$

and

$$\left|\frac{d_k - c_0}{c_1 - c_2}\right| = \left|\frac{\omega_k - \omega^2}{(1 - \omega_k)(1 - \omega^2)}\right| = \frac{|\omega_k|}{|1 - \omega^2|} = \frac{\omega}{1 - \omega^2}\,.$$

Therefore, μ lies in the disc

$$|z - c_0| \le \frac{\omega|c_1 - c_2|}{1 - \omega^2} + \frac{R_1 + \omega R_2}{1 - \omega} = \rho. \tag{A.5}$$

If $|c_0| > \rho,$ then the origin is an exterior point of the disc in (A.5) and Γ is a subset of the circular annulus

$$|c_0| - \rho \le |z| \le |c_0| + \rho. \tag{A.6}$$

Moreover, if

$$\theta = \text{arc sin}\left(\frac{\rho}{|c_0|}\right),$$

then the locus Γ is the sector defined by (A.6) and

$$\text{Arg}c_0 - \theta \le \text{Arg}z \le \theta + \text{Arg}c_0.$$

If $|c_0| \le \rho,$ then $\sigma(f)$ lies in the disc in (A.5). ■

Remark I The locus Γ in the above proof can be a multiple connected set, since for $r_1 > \omega R_2$,

$$|z - d_k| \ge \frac{|\,|\alpha_1 - c_1| - |\omega_k||\alpha_2 - c_2|\,|}{1 + |\omega_k|} \ge \frac{r_1 - \omega R_2}{1 + \omega},$$

and for $R_1 < \omega r_2$,

$$|z - d_k| \ge \frac{\omega r_2 - R_1}{1 + \omega}.$$

Note also that if $\lambda_1 = \lambda_2 = 1$, then $\omega_k = \sqrt[n]{-1}$ and $\omega = |\omega_k| = 1$. In this case, we have

$$|z - d_k| \leq \frac{R_1 + R_2}{|1 - \omega_k|}.$$

Furthermore, for $r_1 > R_2$,

$$|z - d_k| \geq \frac{r_1 - R_2}{2},$$

and for $R_1 \leq r_2$,

$$|z - d_k| \geq \frac{r_2 - R_1}{2}.$$