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On compressions of normal matrices

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Abstract

Let A be a normal matrix and consider the polygon $NR[A] = \{x^*Ax\colon \|x\| = 1\}$. If $\upsilon^*A\upsilon \in \operatorname{int} NR[A]$, a projector matrix P_υ is defined such that $NR[P_\upsilon^* A P_\upsilon]$ is supported by all or some edges of a polygon.

1. Introduction

In this paper we continue the investigation of compressions and dilations of the numerical ranges of matrices [4], paying special attention to normal matrices.

Let $\mathscr{M}_n$ be the algebra of $n \times n$ complex matrices. For a matrix $A \in \mathscr{M}_n$, the set

$$NR[A] = \left\{x^*Ax\colon x \in \mathbb{C}^n, \ \|x\| = 1\right\}$$

is called *numerical range* or *field of values* of A. Extensive background and generalizations of $NR[A]$ are presented in [3] and several ideas in this area are exposed in the survey [1]. We note below some of them:

1. $NR[A]$ is compact and convex set in $\mathbb{C}$.
2. The spectrum $\sigma(A) \subseteq NR[A]$.
3. $NR[P^*AP] = NR[A]$, where P is any $n \times n$ unitary matrix.
4. $NR[H_A] = \operatorname{Re} NR[A]$, where $H_A = (A + A^*)/2$.

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5. If A is normal, $NR[A] = \mathrm{Co}\{\sigma(A)\}$, where Co{ } denotes the convex hull of the set.

Clearly, from $NR[A]$ one can obtain information about the matrix that the spectrum of A alone cannot give. Combining properties **3** and **4**, we obtain

$$\operatorname{Re} NR[A] = NR[H_{P^*AP}], \quad \operatorname{Im} NR[A] = -NR[H_{\mathrm{i}\,P^*AP}]. \tag{1}$$

Moreover, for an $n \times m$ $(m < n)$ matrix P, such that $P^*P = I_m$, we have

$$NR\big[P^*AP\big] \subset NR[A]. \tag{2}$$

The matrix P is called *isometry* and P^*AP is called an *isometric projection* of A. Recently, in [4] it has been proved that

$$NR[A] = \bigcup_P NR\big[P^*AP\big],$$

where the matrix P is an $n \times m$ isometry.

The inclusion relation of $NR[A]$ in (2) leads to the investigation of P such that $NR[P^*AP]$ and $NR[A]$ are "close enough". In [3], a procedure is discussed for determining and plotting the numerical range of a matrix. The general strategy is to calculate many points on the boundary of the numerical range and determine supporting lines at these points. The intersection of the supporting lines is a convex polygonal approximation of the numerical range that contains it. At this point it is remarkable that no relationship is known between matrices for which only their numerical ranges are known. In this direction, in Section 2 we give a description of P such that, for a normal matrix A, the boundary $\partial NR[P^*AP]$ is tangential to $\partial NR[A]$, and we expose a condition for a "satisfactory" approximation. In the same section, we comment on the case when A is Hermitian. Further, in Section 3, we generalize these results by illustrating how $\partial NR[P^*AP]$ is supported by some edges of $\partial NR[A]$.

2. Compression of NR of normal matrices

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ $(k \leqslant n)$ be eigenvalues of a normal matrix $A \in \mathscr{M}_n$ such that $NR[A] = \mathrm{Co}\{\lambda_1, \ldots, \lambda_k\}$; i.e., the remaining eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ belong to the closed polygon $\langle\lambda_1, \lambda_2, \ldots, \lambda_k\rangle$. In what follows, we denote by $x_1, x_2, \ldots, x_k$ a set of orthonormal eigenvectors of A corresponding to $\lambda_1, \ldots, \lambda_k$, and consider the unit vector

$$\upsilon = \sum_{j=1}^{k} c_j x_j; \quad \sum_{j=1}^{k} |c_j|^2 = 1, \quad |c_j| \neq 0.$$

Obviously, the point $\mu = \upsilon^* A \upsilon$ belongs to int $NR[A]$. Regarding $E = \mathrm{span}\{\upsilon\}$ as a subspace of $W = \mathrm{span}\{x_1, x_2, \ldots, x_k\}$ we consider the $n \times (k-1)$ matrix

$$P = [w_1 \;\; w_2 \;\; \cdots \;\; w_{k-1}], \tag{3}$$

where $w_1, w_2, \ldots, w_{k-1}$ is an orthonormal basis of $E_W^{\perp}$. Clearly, $P^*P = I_{k-1}$ and PP^* is an orthogonal projector onto $E_W^{\perp}$.

Proposition 1. *Let the polygon $\langle\lambda_1, \lambda_2, \ldots, \lambda_k\rangle$ be the numerical range of a normal matrix A. If we consider the points*

$$\rho_\tau = a_\tau\mu_{\tau-1} + (1-a_\tau)\lambda_{\tau+1}, \quad \mu_\tau = (1-a_\tau)\mu_{\tau-1} + a_\tau\lambda_{\tau+1}; \quad \tau = 1, \ldots, k-1,$$

where $a_\tau \in [0, 1]$, $\mu_0 = \lambda_1$, then the matrix A is unitarily similar to a matrix Θ with diagonal elements

$$\rho_1, \ldots, \rho_{k-1}, \mu_{k-1}, \lambda_{k+1}, \ldots, \lambda_n.$$

Proof. Denoting by x_i $(i = 1, \ldots, k)$ a set of orthonormal eigenvectors of A corresponding to the vertices λ_i and by

$$\begin{aligned}
y_1 &= \sqrt{a_1}x_1 + \sqrt{1-a_1}x_2,\\
y_2 &= \sqrt{a_2}\left(\sqrt{1-a_1}x_1 - \sqrt{a_1}x_2\right) + \sqrt{1-a_2}x_3 = \sqrt{a_2}\varepsilon_1 + \sqrt{1-a_2}x_3,\\
y_3 &= \sqrt{a_3}\left(\sqrt{1-a_2}\varepsilon_1 - \sqrt{a_2}x_3\right) + \sqrt{1-a_3}x_4 = \sqrt{a_3}\varepsilon_2 + \sqrt{1-a_3}x_4,\\
y_4 &= \sqrt{a_4}\left(\sqrt{1-a_3}\varepsilon_2 - \sqrt{a_3}x_4\right) + \sqrt{1-a_4}x_5 = \sqrt{a_4}\varepsilon_3 + \sqrt{1-a_4}x_5,\\
&\ \ \vdots\\
y_{k-1} &= \sqrt{a_{k-1}}(\cdots) + \sqrt{1-a_{k-1}}x_k = \sqrt{a_{k-1}}\varepsilon_{k-2} + \sqrt{1-a_{k-1}}x_k,\\
y_k &= \sqrt{1-a_{k-1}}\varepsilon_{k-2} - \sqrt{a_{k-1}}x_k,
\end{aligned}$$

then it is implied directly that

$$(\varepsilon_\tau, x_{\tau+2}) = 0, \quad \|\varepsilon_\tau\| = 1,$$

and by induction

$$(\varepsilon_\tau, \varepsilon_s)_{\tau<s} = \sqrt{1-a_{\tau+1}}\sqrt{1-a_{\tau+2}}\cdots\sqrt{1-a_s}, \quad \varepsilon_\tau^* A\varepsilon_\tau = \mu_\tau.$$

Therefore,

$$\|y_\tau\| = 1, \quad (\varepsilon_\tau, y_\tau) = 0 \ \ (\tau = 1, \ldots, k-2), \quad (y_\tau, y_s)_{\tau<s} = 0,$$

and

$$y_\tau^* Ay_\tau = \rho_\tau \ \ (\tau = 1, \ldots, k-1), \quad y_k^* Ay_k = \mu_{k-1}.$$

Hence, for

$$R = [y_1 \ \cdots \ y_k \ x_{k+1} \ \cdots \ x_n]$$

we obtain $R^*R = I$ and $R^*AR = \Theta$. □

Theorem 1. *Let the matrix A be normal and the polygon* $\langle\lambda_1, \lambda_2, \ldots, \lambda_k\rangle = NR[A]$. *If* $\upsilon = \sum_{j=1}^{k} c_j x_j$ *is a unit vector such that* $\upsilon^* A\upsilon = \mu \in \operatorname{int} NR[A]$, *then*

$$NR\big[P^*AP\big] \subset \langle\lambda_1, \lambda_2, \ldots, \lambda_k\rangle,$$

where the matrix P is defined as in (3) *and the boundary* $\partial NR[P^*AP]$ *is tangential to the edges of the polygon at the points*

$$\begin{aligned}\sigma_\tau &= \frac{|c_{\tau+1}|^2\lambda_\tau + |c_\tau|^2\lambda_{\tau+1}}{|c_{\tau+1}|^2 + |c_\tau|^2} \quad (\tau = 1, \ldots, k-1);\\ \sigma_k &= \frac{|c_1|^2\lambda_k + |c_k|^2\lambda_1}{|c_1|^2 + |c_k|^2}.\end{aligned} \tag{4}$$

Proof. It is enough to verify the relationship (4) for $\tau = 1$. We consider the points μ_τ $(\tau = 1, 2, \ldots, k-3)$ of the polygon such that μ_τ lies in the line segment $\langle\mu_{\tau-1}, \lambda_{\tau+1}\rangle$. Here $\mu_0 = \lambda_1$ and the point μ_{k-2} is defined as the intersection of segments $\langle\mu_{k-3}, \lambda_{k-1}\rangle$ and $\langle\mu_{k-1}, \lambda_k\rangle$, where $\mu_{k-1} = \mu$. If ν_τ is the reflection of μ_τ about the midpoint of $\langle\mu_{\tau-1}, \lambda_{\tau+1}\rangle$, we have

$$\nu_\tau + \mu_\tau = \mu_{\tau-1} + \lambda_{\tau+1}; \quad \mu_0 = \lambda_1.$$

Thus, by Proposition 1, there exist orthonormal vectors $y_1, \ldots, y_{k-1}, y$ such that

$$y_\tau^* A y_\tau = \nu_\tau \ \ (\tau = 1, 2, \ldots, k-1), \quad y^* A y = \mu.$$

Since μ and $\nu_\tau \in NR[A]$ are unique convex combinations of $\lambda_1, \ldots, \lambda_k$, the vectors y, y_τ can be written in the form

$$y = \sum_{j=1}^{k} c_j \ \exp(-\mathrm{i}\theta_j) x_j, \quad y_\tau = \sum_{j=1}^{\tau+1} c_{\tau j} x_j.$$

Due to orthonormality of y and y_τ we have

$$\sum_{j=1}^{k} |c_j|^2 = 1, \quad \sum_{j=1}^{\tau+1} |c_{\tau j}|^2 = 1, \quad \sum_{j=1}^{\tau+1} \bar{c}_{\tau j} c_j \ \exp(-\mathrm{i}\theta_j) = 0. \tag{5}$$

Thus, the unit vectors

$$u_\tau = \sum_{j=1}^{\tau+1} c_{\tau j} \ \exp(\mathrm{i}\theta_j) x_j \quad (\tau = 1, \ldots, k-1)$$

belong to $E_W^\perp$ and

$$u_\tau^* A u_\tau = \sum_{j=1}^{\tau+1} |c_{\tau j}|^2 x_j^* A x_j = \sum_{j=1}^{\tau+1} |c_{\tau j}|^2 \lambda_j = \nu_\tau.$$

Denoting by P the orthogonal projection onto $E_W^\perp$, the unit vector ℓ_τ, which is defined by $P\ell_\tau = u_\tau$, corresponds to the point $\nu_\tau \in NR[P^*AP] \subset NR[A]$. For the point $\sigma_1 \equiv \nu_1 \in \langle\lambda_1, \lambda_2\rangle$, by Eq. (5), we have

$$\sigma_1 = |c_{11}|^2\lambda_1 + |c_{12}|^2\lambda_2;$$

$$|c_{11}|^2 + |c_{12}|^2 = 1, \quad \bar{c}_{11}c_1 \ \exp(-\mathrm{i}\,\theta_1) + \bar{c}_{12}c_2 \ \exp(-\mathrm{i}\,\theta_2) = 0.$$

Combining these equations we obtain

$$\sigma_1 = \frac{|c_{12}|^2\,|c_2|^2}{|c_1|^2}\lambda_1 + |c_{12}|^2\lambda_2 = \frac{|c_{12}|^2\big(|c_2|^2\lambda_1 + |c_1|^2\lambda_2\big)}{|c_1|^2}.$$

Since

$$|c_{12}|^2\big(|c_1|^2 + |c_2|^2\big) = |c_{12}|^2\,|c_1|^2 + |c_{11}|^2\,|c_1|^2 = |c_1|^2,$$

we obviously have

$$\sigma_1 = \frac{|c_2|^2\lambda_1 + |c_1|^2\lambda_2}{|c_1|^2 + |c_2|^2}.$$

Hence, beginning with the point $\mu_1 \in \langle\lambda_1, \lambda_2\rangle$, the point $\sigma_1 \in \partial NR[P^*AP] \cap \langle\lambda_1, \lambda_2\rangle$ has been defined. Continuing with the point $\mu_1' \in \langle\lambda_2, \lambda_3\rangle$ by the same process, we identify the *tangent point*

$$\sigma_2 = \frac{|c_3|^2\lambda_2 + |c_2|^2\lambda_3}{|c_3|^2 + |c_2|^2} \in \langle\lambda_2, \lambda_3\rangle \cap \partial NR\big[P^*AP\big].$$

The process ends when we find the rest of the points $\sigma_3, \dots, \sigma_k$ on the edges $\langle\lambda_3, \lambda_4\rangle, \dots, \langle\lambda_k, \lambda_1\rangle$, respectively. □

Evidently, in Theorem 1, the matrix P^*AP is presented so that the boundary $\partial NR[P^*AP]$ is tangential *to all edges* of the polygon $\langle\lambda_1, \lambda_2, \dots, \lambda_k\rangle \equiv NR[A]$ and not only one edge as done by Mirman [5]. Moreover, for $k = 3$, the numerical range $NR[P^*AP]$ is an ellipse tangential to the triangle $\langle\lambda_1, \lambda_2, \lambda_3\rangle$. This has also been remarked by Williams [6, Th. 1], referring to $NR[PP^*APP^*]$.

It is also worth noting in Theorem 1 that the vector υ is a linear combination one of the eigenvectors x_j of A which correspond to λ_j, for any $j = 1, 2, \dots, k$. If one considers the vector

$$\upsilon = c_{11}x_{11} + c_{12}x_{12} + c_2x_2 + \cdots + c_kx_x,$$

where the eigenvectors x_{11}, x_{12} correspond to the same eigenvalue, say λ_1, then $\sigma_1 \equiv \lambda_1$ and λ_1 is an angular point (sharp point) of $\partial NR[P^*AP]$.

In case the vector υ is dependent on fewer than k eigenvectors, then the compression of $NR[A]$, i.e., $NR[P^*AP]$ does not touch all the edges of the polygon. Then $NR[A]$ and $NR[P^*AP]$ are not so closely related as we see in the following.

Proposition 2. *Let A be a normal matrix and $NR[A] = \langle\lambda_1, \dots, \lambda_k\rangle$. If the unit vector $\upsilon = \sum_{i=1}^{s} c_jx_j$, with $2 \leqslant s \leqslant k-1$, corresponds to $\mu = \upsilon^*A\upsilon \in \mathrm{int}\langle\lambda_1, \dots, \lambda_s\rangle$, then*

$$NR\big[P^*AP\big] = \mathrm{Co}\{NR[B], \lambda_{s+1}, \dots, \lambda_k\},$$

where the matrix $B \in \mathscr{M}_{s-1}$ is a direct component of P^*AP and $\partial NR[B]$ is inscribed to the polygon $\langle\lambda_1, \ldots, \lambda_s\rangle$ at the points $\sigma_\tau (\tau = 1, 2, \ldots, s-1)$ in (4).

Proof. Since $\upsilon = c_1x_1 + \cdots + c_sx_s + 0x_{s+1} + \cdots + 0x_k$, the unit eigenvectors $x_{s+1}, \ldots, x_k$ are orthogonal to υ. If $w_1, \ldots, w_{s-1}$ form an orthonormal basis of $E^\perp_{W_1}$, where $W_1 = \text{span}\{x_1, \ldots, x_s\}$, the vectors $w_1, \ldots, w_{s-1}, x_{s+1}, \ldots, x_k$ form an orthonormal basis of $E^\perp_W$. Denoting by

$$\begin{aligned} Q &= \left[x_1 \ \cdots \ x_s \ \vdots \ x_{s+1} \ \cdots \ x_k\right] = [Q_1 \ \ Q_2], \\ P_1 &= [w_1 \ \cdots \ w_{s-1}], \end{aligned} \tag{6}$$

we have

$$Q^*AQ = \text{diag}(\lambda_1, \ldots, \lambda_s, \lambda_{s+1}, \ldots, \lambda_k) = \text{diag}(D_1, D_2)$$

and

$$P^*AP = B \oplus \text{diag}(\lambda_{s+1}, \ldots, \lambda_k),$$

where

$$P = [P_1 \ \ Q_2], \quad B = P_1^* Q_1 D_1 Q_1^* P_1.$$

Hence,

$$NR\big[P^*AP\big] = \text{Co}\{NR[B], \lambda_{s+1}, \ldots, \lambda_k\}. \tag{7}$$

Moreover, the singular matrix $A_1 = Q_1 D_1 Q_1^* \in \mathscr{M}_n$ is normal, $\sigma(A_1) = \{\lambda_1, \ldots, \lambda_s, 0\}$ and due to $NR[B] \equiv NR[P_1^* A_1 P_1]$, by Theorem 1, $\partial NR[B]$ is tangential to the polygon $\langle\lambda_1, \ldots, \lambda_s\rangle$ at the points σ_τ of the edges $\langle\lambda_\tau, \lambda_{\tau+1}\rangle$, $\tau = 1, 2, \ldots, s-1$ (as has been presented in (4)). □

Remark 1. In Proposition 2, if $\upsilon = c_1x_1 + 0x_2 + c_3x_3 + \cdots + c_sx_s$, with $c_i \neq 0$, then $\partial NR[B]$ is supported by the edges of the polygon $\langle\lambda_1, \lambda_3, \ldots, \lambda_s\rangle$. We have that

$$\frac{|c_1|^2\lambda_3 + |c_3|^2\lambda_1}{|c_1|^2 + |c_3|^2} \in \langle\lambda_1, \lambda_3\rangle \cap \partial NR[B].$$

Remark 2. If you consider $\{w_i\}$ as a basis of $E^\perp$ instead of $E^\perp_{W_1}$, then

$$NR\big[P^*AP\big] = \text{Co}\{NR[B], \lambda_{s+1}, \ldots, \lambda_n\},$$

recalling that $\lambda_{k+1}, \ldots, \lambda_n \in \langle\lambda_1, \ldots, \lambda_k\rangle$.

Remark 3. Let the eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ of the normal matrix A be interior points of the polygon $\langle\lambda_1, \ldots, \lambda_k\rangle$ and let the unit vector $\upsilon \in \text{span}\{x_{k+1}, \ldots, x_n\}$, for $\mu = \upsilon^*A\upsilon$, where x_i are the eigenvectors of A corresponding to λ_i. If the columns of

$$P = \left[x_1 \ x_2 \ \cdots \ x_k \ \vdots \ y_1 \ \cdots \ y_{n-k-1}\right] = [X \ \ Y]$$

form an orthonormal basis of $\text{span}\{\upsilon\}^\perp$, then

$$P^*AP = \begin{bmatrix} X^* \\ Y^* \end{bmatrix} A[X \quad Y] = \begin{bmatrix} \mathrm{diag}(\lambda_1, \ldots, \lambda_k) & O \\ O & Y^*AY \end{bmatrix}.$$

Moreover, due to

$$NR[Y^*AY] \subset \langle \lambda_{k+1}, \ldots, \lambda_n \rangle,$$

we have

$$NR\big[P^*AP\big] = \langle \lambda_1, \ldots, \lambda_k \rangle.$$

A procedure for the compression of $NR[A]$ is presented in Appendix A.

Example 1. Consider the normal matrix

$$A = \mathrm{diag}\left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \right)$$

with

$$\sigma(A) = \left\{ \sqrt{3}\mathrm{i}, -\sqrt{3}\mathrm{i}, 3, 1 + \sqrt{3}\mathrm{i}, 1 - \sqrt{3}\mathrm{i}, 1 \right\}.$$

The set $\mathrm{Co}\{\sigma(A)\}$ is the polygon

$$\langle \lambda_1, \ldots, \lambda_5 \rangle = \left\langle \sqrt{3}\mathrm{i}, -\sqrt{3}\mathrm{i}, 1 - \sqrt{3}\mathrm{i}, 3, 1 + \sqrt{3}\mathrm{i} \right\rangle.$$

For the unit vectors

$$\upsilon = \frac{\sqrt{3}\mathrm{i}}{4} x_1 + \frac{1}{2} x_2 - \frac{\sqrt{2}\mathrm{i}}{4} x_3 - \frac{\sqrt{3}\mathrm{i}}{4} x_4 + \frac{\sqrt{3} + \mathrm{i}}{4} x_5,$$

$$\hat{\upsilon} = \frac{1 + \sqrt{3}\mathrm{i}}{3} x_1 - \frac{1}{3} x_3 + \frac{\sqrt{3} + \mathrm{i}}{3} x_4,$$

the points $\upsilon^* A \upsilon$ and $\hat{\upsilon}^* A \hat{\upsilon}$ belong to $NR[A]$. Then the isometric matrix in (3) is

$$P = \begin{bmatrix} -0.6224 & -0.2446 - 0.0470\mathrm{i} & -0.5346 + 0.4434\mathrm{i} & -0.1022 + 0.0812\mathrm{i} \\ 0.7640 - 0.0663\mathrm{i} & -0.1315 + 0.0122\mathrm{i} & -0.4588 + 0.3511\mathrm{i} & -0.0076 + 0.1229\mathrm{i} \\ -0.1416 + 0.0663\mathrm{i} & 0.3762 + 0.0349\mathrm{i} & -0.3410 - 0.1095\mathrm{i} & 0.4040 + 0.1136\mathrm{i} \\ 0 & 0.3201 + 0.3960\mathrm{i} & -0.1211 + 0.0621\mathrm{i} & -0.5676 + 0.0244\mathrm{i} \\ 0 & -0.5030 + 0.0793\mathrm{i} & 0.0067 - 0.1359\mathrm{i} & 0.4547 + 0.3407\mathrm{i} \\ 0 & 0.1828 - 0.4752\mathrm{i} & 0.1143 + 0.0738\mathrm{i} & 0.1129 - 0.3651\mathrm{i} \end{bmatrix}.$$

The boundary $\partial NR[P^*AP]$ is the closed curve in Fig. 1(a), which is tangential to the edges of the polygon $\langle \lambda_1, \ldots, \lambda_5 \rangle$ at the points defined by (4),

$$\sigma_1 = \frac{\sqrt{3}\mathrm{i}}{7}, \quad \sigma_2 = \frac{2 - 3\sqrt{3}\mathrm{i}}{3}, \quad \sigma_3 = \frac{9 - 3\sqrt{3}\mathrm{i}}{5}, \quad \sigma_4 = \frac{15 + 3\sqrt{3}\mathrm{i}}{7},$$

$$\sigma_5 = \frac{3 + 7\sqrt{3}\mathrm{i}}{7}.$$

On the other hand, by (6) we have

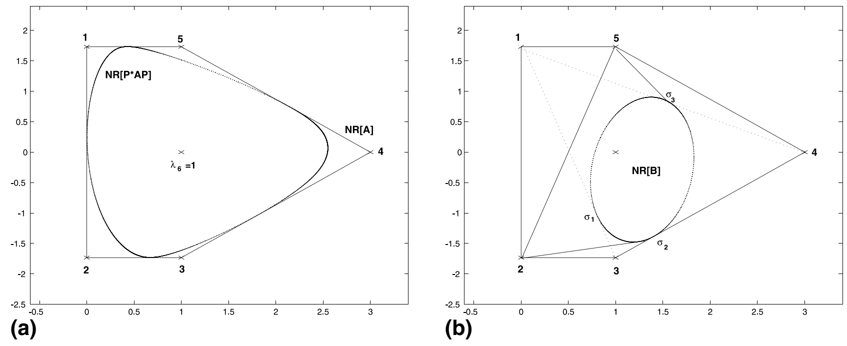

Fig. 1.

$$P_1 = \begin{bmatrix} -0.2582 & 0.0081 - 0.4208\mathrm{i} \\ 0.1291 - 0.2236\mathrm{i} & -0.3647 + 0.0446\mathrm{i} \\ 0.1291 + 0.2236\mathrm{i} & -0.5814 - 0.5109\mathrm{i} \\ 0.4245 + 0.2941\mathrm{i} & 0.1674 + 0.0399\mathrm{i} \\ -0.4669 + 0.2206\mathrm{i} & -0.1183 + 0.1251\mathrm{i} \\ 0.0424 - 0.5147\mathrm{i} & -0.0492 - 0.1650\mathrm{i} \end{bmatrix}, \quad P = [P_1 \;\; x_2 \;\; x_5]$$

and

$$B = \begin{bmatrix} 0.8000 - 1.0392\mathrm{i} & -0.0873 - 0.9575\mathrm{i} \\ 0.5841 - 0.7637\mathrm{i} & 1.7556 + 0.4619\mathrm{i} \end{bmatrix}.$$

In Fig. 1(b), the boundary $\partial NR[B]$ is tangential to the segments $\langle \lambda_1, \lambda_3 \rangle$, $\langle \lambda_3, \lambda_4 \rangle$ and $\langle \lambda_4, \lambda_1 \rangle$ at the points

$$\hat{\sigma}_1 = \frac{4 - 3\sqrt{3}\mathrm{i}}{5}, \quad \hat{\sigma}_2 = \frac{7 - 4\sqrt{3}\mathrm{i}}{5}, \quad \hat{\sigma}_3 = \frac{3 + \sqrt{3}\mathrm{i}}{2}.$$

Hence by (7), $NR[P^*AP] = \mathrm{Co}\{NR[B], \lambda_2, \lambda_5\}$.

Remark 4. If $\lambda_1 = a_1 + \mathrm{i}\beta_1$ and $\lambda_2 = a_2 + \mathrm{i}\beta_2$, then $(\beta_2 - \beta_1)x + (a_2 - a_1)y - (a_1\beta_2 - \beta_1 a_2) = 0$ is the real equation of the segment $\langle \lambda_1, \lambda_2 \rangle$. Denoting by $u = c_1x_1 + c_2x_2$ the unit vector which corresponds to the point $\sigma_1 = \langle \lambda_1, \lambda_2 \rangle \cap NR[P^*AP]$ in Theorem 1 and letting $A = A_1 + \mathrm{i}\, A_2$, where A_1, A_2 are Hermitian matrices, we have that *$a_1\beta_2 - \beta_1 a_2$ is not simple eigenvalue of the Hermitian matrix $M = (\beta_2 - \beta_1)H_1 + (a_1 - a_2)H_2$, where $H_1 = P^*A_1P$ and $H_2 = P^*A_2P$.*

In fact, using the statement for normal matrices $Ax_j = \lambda_j x_j \Longrightarrow A_1x_j = a_jx_j$, $A_2x_j = \beta_jx_j$ $(j = 1, 2)$, the solution of equation $P\ell = u$ defines an eigenvector of M:

$$\begin{aligned} M\ell &= (\beta_2 - \beta_1)P^*A_1u + (a_1 - a_2)P^*A_2u \\ &= P^*[(\beta_2 - \beta_1)A_1(c_1x_1 + c_2x_2) + (a_1 - a_2)A_2(c_1x_1 + c_2x_2)] \end{aligned}$$

$$= P^*[(\beta_2 - \beta_1)(c_1 a_1 x_1 + c_2 a_2 x_2) + (a_1 - a_2)(c_1 \beta_1 x_1 + c_2 \beta_2 x_2)]$$
$$= (a_1\beta_2 - a_2\beta_1) P^*(c_1 x_1 + c_2 x_2)$$
$$= (a_1\beta_2 - a_2\beta_1)\ell.$$

Moreover, the solutions of equations $P\xi_j = x_j$ $(j = 1, 2)$ define another eigenvector of M corresponding to the same eigenvalue $a_1\beta_2 - a_2\beta_1$, i.e.,

$$\begin{aligned} &M(c_1\xi_1 - c_2\xi_2) \\ &\quad = (\beta_2 - \beta_1)P^* A_1(c_1 x_1 - c_2 x_2) + (a_1 - a_2)P^* A_2(c_1 x_1 - c_2 x_2) \\ &\quad = (a_1\beta_2 - a_2\beta_1)P^*(c_1 x_1 - c_2 x_2) \\ &\quad = (a_1\beta_2 - a_2\beta_1)(c_1\xi_1 - c_2\xi_2). \end{aligned}$$

Hence, by this remark we conclude that the formula [2, Th. 1.1] for the radius of curvature of $NR[P^*AP]$ at its boundary points $\sigma_i = \ell^* P^* A P \ell$ in Theorem 1, cannot be used.

Remark 5. The role of matrix P in Theorem 1 provides the tinder for a simplified formulation of a non-orthogonal basis $\{y_1, \ldots, y_{k-1}\}$ of $E_W^{\perp}$. If $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$, setting

$$y_t = f_t e_t + g_t e_{t+1} \in \mathbb{C}^n,$$

by the equations $(\upsilon, y_t) = 0$, $t = 1, 2, \ldots, k-1$ $(k \leqslant n)$, we find the relationships

$$\upsilon_t = (-1)^{t-1} c \bar{f}_1 \bar{f}_2 \cdots \bar{f}_{t-1} \bar{g}_t \cdots \bar{g}_{k-1}; \quad t = 1, \ldots, k, \quad f_0 = 1,$$

for the first k coordinates of υ. Therefore,

$$\frac{g_t}{f_t} = -\frac{\bar{\upsilon}_t}{\bar{\upsilon}_{t+1}}$$

and due to $\|y_t\| = 1$ we have

$$\begin{aligned} &|f_t|^2 = \frac{|\upsilon_{t+1}|^2}{|\upsilon_t|^2 + |\upsilon_{t+1}|^2}, \quad |g_t|^2 = \frac{|\upsilon_t|^2}{|\upsilon_t|^2 + |\upsilon_{t+1}|^2}, \\ &\text{Arg } g_t - \text{Arg } f_t = \pi - \text{Arg } \upsilon_t + \text{Arg } \upsilon_{t+1}. \end{aligned} \tag{8}$$

Applying Gram–Schmidt orthonormalization to the basis $\{y_1, \ldots, y_{k-1}\}$, we find an $n \times (k-1)$ matrix R, such that $R^* R = I_{k-1}$. For $\hat{P} = RR^*$,

$$NR[\hat{P} \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\hat{P}] \subset \langle \lambda_1, \ldots, \lambda_k \rangle = NR[A].$$

Moreover, after some algebraic manipulations we obtain

$$\begin{aligned} \hat{p}_t &= y_t^* \hat{P} \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\hat{P} y_t \\ &= y_t^* \operatorname{diag}(\lambda_1, \ldots, \lambda_n) y_t \\ &= |f_t|^2 \lambda_t + |g_t|^2 \lambda_{t+1} \end{aligned}$$

and the vector

$$y_k = \left[f_1 f_2 \cdots f_{k-1}, 0, \ldots, 0, (-1)^k g_1 g_2 \cdots g_{k-1}, 0, \ldots, 0\right]^{\mathrm{T}}$$

corresponds to the point

$$\begin{aligned}\hat{p}_k &= y_k^* \hat{P} \operatorname{diag}(\lambda_1, \dots, \lambda_n) \hat{P} y_k \\ &= \frac{1}{\|y_k\|^2}\left(|f_1 \cdots f_{k-1}|^2 \lambda_1 + |g_1 \cdots g_{k-1}|^2 \lambda_k\right).\end{aligned}$$

Obviously, $\hat{p}_1, \dots, \hat{p}_k$ are tangent points of $\partial NR[A]$ and $\partial NR[\hat{P} \operatorname{diag}(\lambda_1, \dots, \lambda_n)\hat{P}]$.

For $n = k = 3$, Mirman [5] also mentions the special form of y_1, y_2, y_3, and using these vectors he computes each tangent point.

Example 2. Consider the matrix A in Example 1 and the unit vector

$$\upsilon = \frac{\sqrt{3}}{3}x_1 + \frac{\sqrt{2}}{3}x_2 - \frac{1}{3}x_3 + \frac{1}{3}x_4 + \frac{\sqrt{2}}{3}x_5 = \begin{bmatrix} 0.1427 - 0.0610\mathrm{i} \\ 0.7399 + 0.0261\mathrm{i} \\ -0.3053 + 0.0348\mathrm{i} \\ -0.0797 \\ 0.0399 + 0.4024\mathrm{i} \\ 0.0399 - 0.4024\mathrm{i} \end{bmatrix}.$$

Using Eqs. (8) we define the vectors:

$$y_1 = [0.9787, -0.1857 - 0.0872\mathrm{i}, 0, 0, 0, 0]^{\mathrm{T}},$$

$$y_2 = [0, 0.3833, 0.9134 - 0.1368\mathrm{i}, 0, 0, 0]^{\mathrm{T}},$$

$$y_3 = [0, 0, 0.2511, -0.9617 - 0.1096\mathrm{i}, 0, 0]^{\mathrm{T}},$$

$$y_4 = [0, 0, 0, 0.9811, 0.0187 + 0.1924\mathrm{i}, 0]^{\mathrm{T}},$$

and consequently

$$y_5 = [0.0924, 0, 0, 0, 0.0107 - 0.0338\mathrm{i}, 0]^{\mathrm{T}}.$$

Therefore, the isometric projector matrix is

$$\hat{P} = \begin{bmatrix} 0.9712 & -0.1243 + 0.0584\mathrm{i} & 0.0546 - 0.0163\mathrm{i} & 0.0136 - 0.0058\mathrm{i} & 0.0227 + 0.0715\mathrm{i} & 0 \\ -0.1243 - 0.0584\mathrm{i} & 0.3449 & 0.2689 + 0.0403\mathrm{i} & 0.0705 + 0.0025\mathrm{i} & -0.0471 + 0.3549\mathrm{i} & 0 \\ 0.0546 + 0.0163\mathrm{i} & 0.2689 - 0.0403\mathrm{i} & 0.8872 & -0.0291 + 0.0033\mathrm{i} & -0.0025 - 0.1486\mathrm{i} & 0 \\ 0.00136 + 0.0058\mathrm{i} & 0.0705 - 0.0025\mathrm{i} & -0.0291 - 0.0033\mathrm{i} & 0.9924 & 0.0037 - 0.0384\mathrm{i} & 0 \\ 0.0227 - 0.0715\mathrm{i} & -0.0472 - 0.3549\mathrm{i} & -0.0025 + 0.1486\mathrm{i} & 0.0037 + 0.0384\mathrm{i} & 0.8043 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the points $\hat{p}_1 = 1.5862\mathrm{i}$, $\hat{p}_2 = 0.8531 - \sqrt{3}\mathrm{i}$, $\hat{p}_3 = 2.8737 - 0.1092\mathrm{i}$, $\hat{p}_4 = 2.9250 + 0.0647\mathrm{i}$ and $\hat{p}_5 = 0.1283 + \sqrt{3}\mathrm{i}$ belong to

$$\partial NR\left[\hat{P} \operatorname{diag}\left(\sqrt{3}\mathrm{i}, -\sqrt{3}\mathrm{i}, 1 - \sqrt{3}\mathrm{i}, 3, 1 + \sqrt{3}\mathrm{i}, 1\right)\hat{P}\right] \cap \partial NR[A].$$

Remark 6 (*the Hermitian case*). Let the matrix A be Hermitian with $k \geqslant 3$ distinct eigenvalues $\lambda_{\min} < \lambda_1 \leqslant \cdots \leqslant \lambda_{k-2} < \lambda_{\max}$ and $x_{\min}, x_1, \dots, x_{\max}$ the corresponding orthonormal eigenvectors. Then by Theorem 1, we have the following cases:

1. Consider the unit vector $\upsilon = c_1 x_{\min} + c_2 x_{\max}$. For

$$P = \left[\frac{w_1}{\|w_1\|} \ \ x_1 \ \ x_2 \ \ \cdots \ \ x_{k-2} \right]; \quad w_1 = x_{\min} - \frac{\bar{c}_1}{\bar{c}_2} x_{\max}$$

the numerical range of the Hermitian matrix P^*AP is the line segment $\langle \sigma_1, \sigma_2 \rangle$, where $\sigma_1 = \min\{\varepsilon, \lambda_1\}$, $\sigma_2 = \max\{\varepsilon, \lambda_{k-2}\}$, and

$$\varepsilon = \frac{w_1^* A w_1}{\|w_1\|^2} = |c_2|^2 \lambda_{\min} + |c_1|^2 \lambda_{\max}.$$

2. Consider $\upsilon = c_1 x_{\min} + c_2 x_j$; $j \in \{1, 2, \ldots, k-2\}$. For

$$P = \left[\frac{\hat{w}_1}{\|\hat{w}_1\|} \ \ x_1 \ \ \cdots \ \ x_{j-1} \ \ x_{j+1} \ \ \cdots \ \ x_{k-2} \ \ x_{\max} \right]; \quad \hat{w}_1 = x_{\min} - \frac{\bar{c}_1}{\bar{c}_2} x_j,$$

then $NR[P^*AP] = \langle \hat{\varepsilon}, \lambda_{\max} \rangle$, where $\hat{\varepsilon} = \min\{|c_1|^2 \lambda_j + |c_2|^2 \lambda_{\min}, \lambda_1\}$.

3. Let $\upsilon = c_1 x_{\min} + c_2 x_j + c_3 x_{\max}$. For

$$P = \left[\frac{w_1}{\|w_1\|} \ \ \frac{w_2}{\|w_2\|} \ \ x_1 \ \ \cdots \ \ x_{j-1} \ \ x_{j+1} \ \ \cdots \ \ x_{k-2} \right]$$

with

$$w_1 = x_{\min} - \frac{\bar{c}_1}{\bar{c}_2} x_j, \quad w_2 = x_{\max} - \frac{\bar{c}_3}{\bar{c}_2} x_j,$$

we obtain $NR[P^*AP] = \langle \sigma_1, \sigma_2 \rangle$, where now σ_1 and σ_2 are the *minimum* and *maximum* of the set $\{\varepsilon_1, \varepsilon_2, \lambda_1, \lambda_{k-2}\}$, with

$$\varepsilon_1 = \frac{w_1^* A w_1}{\|w_1\|^2} = \frac{|c_1|^2 \lambda_j + |c_2|^2 \lambda_{\min}}{|c_1|^2 + |c_2|^2},$$

$$\varepsilon_2 = \frac{w_2^* A w_2}{\|w_2\|^2} = \frac{|c_2|^2 \lambda_j + |c_3|^2 \lambda_{\max}}{|c_2|^2 + |c_3|^2}.$$

3. Generalizations

In what follows, we consider the linearly independent unit vectors

$$\upsilon_j = \sum_{i=1}^{k} c_{ij} x_i \in W, \quad j = 1, \ldots, k-1,$$

which correspond to the points $\mu_1, \ldots, \mu_{k-1}$ of $\langle \lambda_1, \ldots, \lambda_k \rangle = NR[A]$, i.e., $\mu_j = \upsilon_j^* A \upsilon_j$. For $E = \text{span}\{\upsilon_1, \ldots, \upsilon_{k-1}\}$, the subspace $E_W^{\perp}$ is spanned by a vector w. Noting that

$$w = \sum_{j=1}^{k} \theta_j x_j,$$

the parameters θ_j are defined by the equations $(w, \upsilon_j) = 0$ $(j = 1, \dots, k-1)$.

Proposition 3. *If* $P = [w \ \ x_{k+1} \ \ \cdots \ \ x_n]$, *the compressed matrix* $P^*AP \in \mathscr{M}_{n-k+1}$ *is normal with eigenvalues*

$$\lambda_0 = \sum_{j=1}^{k} |\theta_j|^2 \lambda_j, \quad \lambda_{k+1}, \dots, \lambda_n.$$

Proof. Clearly, $P^*w = e_1$. Since $PP^*w = w$, e_1 is an eigenvector of P^*AP as it is implied by

$$e_1^* P^* A P e_1 = w^* P P^* A P P^* w = w^* A w = \sum_{j=1}^{k} |\theta_j|^2 \lambda_j = \lambda_0.$$

Moreover, due to $P^* x_\sigma = e_\sigma$ $(\sigma = k+1, \dots, n)$, we have

$$e_\sigma^* P^* A P e_\sigma = x_\sigma^* P P^* A P P^* x_\sigma = x_\sigma^* A x_\sigma = \lambda_\sigma.$$

Hence, P^*AP is normal with the indicated spectrum. □

Obviously, for $\upsilon_1 = x_1, \dots, \upsilon_{k-1} = x_{k-1}$, we have that $E_W^\perp = \operatorname{span}\{x_k\}$ and λ_k, $\lambda_{k+1}, \dots, \lambda_n$ are eigenvalues of the normal matrix P^*AP.

Theorem 2. *Let* $\langle \lambda_1, \dots, \lambda_k \rangle$ *be the numerical range of the normal matrix A and suppose that the unit vectors*

$$\upsilon_1 = \sum_{j=1}^{k} c_j x_j, \quad \upsilon_2 = \sum_{j=1}^{s} d_j x_j \ \ (s < k)$$

correspond to the points μ_1 *and* μ_2 *of* $NR[A]$. *If the columns of* $P = [w_1 \ \ \dots \ \ w_{k-2}]$ *are an orthonormal basis of* $\operatorname{span}\{\upsilon_1, \upsilon_2\}_W^\perp$, *then*:

(I) $c_i d_{i+1} - c_{i+1} d_i = 0$

$$\Longrightarrow \quad \sigma_{i,i+1} = \frac{|c_i|^2 \lambda_{i+1} + |c_{i+1}|^2 \lambda_i}{|c_i|^2 + |c_{i+1}|^2} \in \partial NR[A] \cap \partial NR\big[P^*AP\big], \quad 1 \leqslant i \leqslant s-1.$$

(II) $$\operatorname{rank}\begin{bmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{bmatrix} = 1 \quad \Longrightarrow \quad NR\big[P^*AP\big] \subseteq \operatorname{Co}\{NR[B], \lambda_{s+1}, \dots, \lambda_k\}.$$

Proof. (I) Let $d_{i+1} \neq 0$. By assumption,

$$w = x_i - \frac{\bar{d}_i}{\bar{d}_{i+1}} x_{i+1} = x_i - \frac{\bar{c}_i}{\bar{c}_{i+1}} x_{i+1} \in \operatorname{span}\{\upsilon_1, \upsilon_2\}_W^\perp.$$

Hence, according to Proposition 2, the point

$$\sigma_{i,i+1} = \frac{1}{\|w\|^2} w^* A w = \frac{|c_i|^2 \lambda_{i+1} + |c_{i+1}|^2 \lambda_i}{|c_i|^2 + |c_{i+1}|^2}$$

is a tangent point of the boundaries $\partial NR[P^*AP]$ and $\partial NR[A]$.

(II) As in case (I), the vectors

$$x_1 - \frac{\bar{c}_1}{\bar{c}_{i+1}} x_{i+1}, \ldots, x_s - \frac{\bar{c}_s}{\bar{c}_{i+1}} x_{i+1}, \qquad x_{s+1} - \frac{\bar{c}_{s+1}}{\bar{c}_m} x_m, \ldots, x_k - \frac{\bar{c}_k}{\bar{c}_m} x_m,$$

for $c_m \neq 0$ $(m > s)$, are orthogonal to υ_1, υ_2. Orthonormalizing these vectors, the matrix

$$P = \left[w_1 \quad \cdots \quad w_{s-1} \ \vdots \ w_s \quad \cdots \quad w_{k-2} \right] = [P_1 \quad P_2]$$

is defined and due to the orthogonality of the vectors

$$\left\{ x_1 - \frac{\bar{c}_1}{\bar{c}_{i+1}} x_{i+1}, \ldots, x_s - \frac{\bar{c}_s}{\bar{c}_{i+1}} x_{i+1} \right\}$$

and

$$\left\{ x_{s+1} - \frac{\bar{c}_{s+1}}{\bar{c}_m} x_m, \ldots, x_k - \frac{\bar{c}_k}{\bar{c}_m} x_m \right\}$$

we have

$$P_1 = [x_1 \quad x_2 \quad \cdots \quad x_s] Q_1, \qquad P_2 = [x_{s+1} \quad x_{s+2} \quad \cdots \quad x_k] Q_2.$$

Therefore,

$$\begin{aligned}
P^*AP &= \begin{bmatrix} P_1^* \\ P_2^* \end{bmatrix} [x_1 \quad \cdots \quad x_n] \operatorname{diag}(\lambda_1, \ldots, \lambda_n) [x_1 \quad \cdots \quad x_n]^* [P_1 \quad P_2] \\
&= \begin{bmatrix} P_1^*[x_1 \quad \cdots \quad x_s] & P_1^*[x_{s+1} \quad \cdots \quad x_n] \\ P_2^*[x_1 \quad \cdots \quad x_s] & P_2^*[x_{s+1} \quad \cdots \quad x_n] \end{bmatrix} \\
&\quad \times \begin{bmatrix} \operatorname{diag}(\lambda_1, \ldots, \lambda_s) & O \\ O & \operatorname{diag}(\lambda_{s+1}, \ldots, \lambda_n) \end{bmatrix} \\
&\quad \times \begin{bmatrix} \begin{bmatrix} x_1^* \\ \vdots \\ x_s^* \end{bmatrix} P_1 & \begin{bmatrix} x_1^* \\ \vdots \\ x_s^* \end{bmatrix} P_2 \\ \begin{bmatrix} x_{s+1}^* \\ \vdots \\ x_n^* \end{bmatrix} P_1 & \begin{bmatrix} x_{s+1}^* \\ \vdots \\ x_n^* \end{bmatrix} P_2 \end{bmatrix} \\
&= \begin{bmatrix} Q_1^* & O \\ O & [Q_2^* \quad O] \end{bmatrix} \begin{bmatrix} \operatorname{diag}(\lambda_1, \ldots, \lambda_s) & O \\ O & \operatorname{diag}(\lambda_{s+1}, \ldots, \lambda_n) \end{bmatrix} \\
&\quad \times \begin{bmatrix} Q_1 & O \\ O & \begin{bmatrix} Q_2 \\ O \end{bmatrix} \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} Q_1^* \operatorname{diag}(\lambda_1, \dots, \lambda_s) Q_1 & O \\ O & Q_2^* \operatorname{diag}(\lambda_{s+1}, \dots, \lambda_k) Q_2 \end{bmatrix}$$
$$= B \oplus \Gamma.$$

Consequently,

$$NR\big[P^* A P\big] = \mathrm{Co}\{NR[B] \cup NR[\Gamma]\}.$$

Clearly, the vectors υ_1, υ_2 are related by $\upsilon_1 = c\upsilon_2 + \hat{\upsilon}_2$, where $\hat{\upsilon}_2 = \sum_{j=s+1}^{k} c_j x_j$. By Proposition 2, for $\upsilon = \upsilon_2$ the boundary $\partial NR[B]$ is tangential to the polygon $\langle \lambda_1, \dots, \lambda_s \rangle$ and for $\upsilon = \hat{\upsilon}_{20} = \hat{\upsilon}_2 / \|\hat{\upsilon}_2\|$, the boundary $\partial NR[\Gamma]$ is tangential to the polygon $\langle \lambda_{s+1}, \dots, \lambda_k \rangle$. Evidently,

$$NR\big[P^* A P\big] \subset \mathrm{Co}\{NR[B], \lambda_{s+1}, \dots, \lambda_k\}. \qquad \square$$

If we consider $R = [P \;\; x_{k+1} \;\; \cdots \;\; x_n]$, then $NR[R^* A R] = \mathrm{Co}\{NR[B] \cup NR[\Gamma], \lambda_{k+1}, \dots, \lambda_n\}$.

Example 3. Consider the matrix

$$A = \operatorname{diag}(4 + 5\mathrm{i}, 2 + 4\mathrm{i}, -1, 1 - 2\mathrm{i}, 5 - \mathrm{i}, 6 + \mathrm{i}, 6 + 3\mathrm{i}, -0.5\mathrm{i})$$

and the unit vectors

$$\upsilon_1 = \frac{\sqrt{3}}{4} e_1 + \frac{\sqrt{2}}{4} e_2 + \frac{1}{2} e_3 + \frac{1}{4} e_4 + \frac{\sqrt{3}}{4} e_5 + \frac{\sqrt{2}}{4} e_6 - \frac{1}{4} e_7,$$

$$\upsilon_2 = \frac{\sqrt{3}}{3} e_1 + \frac{\sqrt{2}}{3} e_2 + \frac{2}{3} e_3.$$

Clearly, $\partial NR[A] = \langle 4 + 5\mathrm{i}, 2 + 4\mathrm{i}, -1, 1 - 2\mathrm{i}, 5 - \mathrm{i}, 6 + \mathrm{i}, 6 + 3\mathrm{i} \rangle$, i.e., $-0.5\mathrm{i} \in \operatorname{int} NR[A]$. According to (II) – Theorem 2, orthonormalizing the vectors

$$e_1 - \frac{\sqrt{3}}{\sqrt{2}} e_2, \quad e_3 - \sqrt{2} e_2, \quad e_4 - \frac{\sqrt{3}}{3} e_5, \quad e_6 - \frac{\sqrt{2}}{\sqrt{3}} e_5, \quad e_7 + \frac{\sqrt{3}}{3} e_5$$

we have

$$P = \left[\begin{array}{cc|ccc} -0.6325 & -0.5164 & 0 & 0 & 0 \\ 0.7746 & -0.4216 & 0 & 0 & 0 \\ 0 & 0.7454 & 0 & 0 & 0 \\ 0 & 0 & -0.8660 & -0.2887 & -0.1543 \\ 0 & 0 & 0.5000 & -0.5000 & -0.2673 \\ 0 & 0 & 0 & 0.8165 & -0.2182 \\ 0 & 0 & 0 & 0 & -0.9258 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [P_1 \;\; P_2].$$

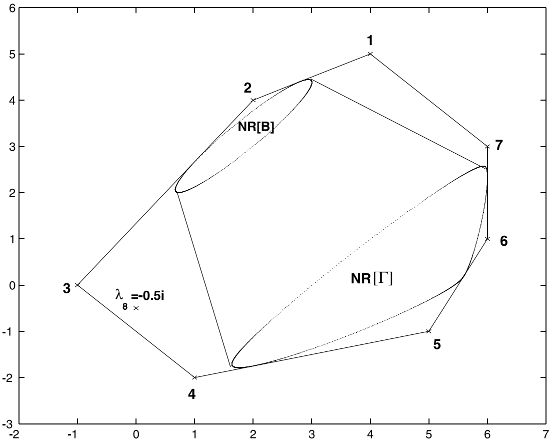

Fig. 2.

Hence, for

$$Q_1 = [e_1 \;\; e_2 \;\; e_3]^* P_1, \qquad Q_2 = [e_4 \;\; e_5 \;\; e_6 \;\; e_7]^* P_2$$

we find

$$\begin{aligned} B &= Q_1^* \operatorname{diag}(4+5\mathrm{i}, 2+4\mathrm{i}, -1) Q_1 \\ &= \begin{bmatrix} 2.8000 + 4.4000\mathrm{i} & 0.6532 + 0.3266\mathrm{i} \\ 0.6532 + 0.3266\mathrm{i} & 0.8667 + 2.0444\mathrm{i} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \Gamma &= Q_2^* \operatorname{diag}(1-2\mathrm{i}, 5-\mathrm{i}, 6+\mathrm{i}, 6+3\mathrm{i}) Q_2 \\ &= \begin{bmatrix} 2.0000 - 1.7500\mathrm{i} & -1.0000 - 0.2500\mathrm{i} & -0.5345 - 0.1336\mathrm{i} \\ -1.0000 - 0.2500\mathrm{i} & 5.3333 + 0.2500\mathrm{i} & -0.3563 - 0.4009\mathrm{i} \\ -0.5345 - 0.1336\mathrm{i} & -0.3563 - 0.4009\mathrm{i} & 5.8095 + 2.5000\mathrm{i} \end{bmatrix}. \end{aligned}$$

In Fig. 2, the boundary $\partial NR[B]$ is tangential to the edges $\langle \lambda_1, \lambda_2 \rangle$ and $\langle \lambda_2, \lambda_3 \rangle$ at the points

$$\sigma_1 = \frac{14+22\mathrm{i}}{5}, \quad \sigma_2 = \frac{3+8\mathrm{i}}{3},$$

and the boundary $\partial NR[\Gamma]$ is tangential to the edges $\langle \lambda_4, \lambda_5 \rangle$, $\langle \lambda_5, \lambda_6 \rangle$ and $\langle \lambda_6, \lambda_7 \rangle$ at the points

$$\sigma_4 = \frac{8-7\mathrm{i}}{4}, \quad \sigma_5 = \frac{28+\mathrm{i}}{5}, \quad \sigma_6 = \frac{18+7\mathrm{i}}{3}.$$

Hence,

$$NR\big[P^* A P\big] = \mathrm{Co}\{NR[B] \cup NR[\Gamma]\}$$

and

$$\partial NR\big[P^*AP\big] \cap \partial NR[A] = \partial NR[A] \cap \{\partial NR[B] \cup \partial NR[\Gamma]\}.$$

Appendix A

The following algorithm refers to the construction of an isometry matrix P such that $NR[P^*AP]$ is inscribed to a polygon.

Step 1. Introduce the normal matrix $A \in \mathscr{M}_n$.

Step 2. Illustrate the numerical range $NR[A]$.

Step 3. Calculate the eigenvectors x_i, $i = 1, \ldots, n$, of A and choose those corresponding to the k vertices of the polygon.

Step 4. Introduce the vector $u = [c_1 \;\; c_2 \;\; \cdots \;\; c_k]^{\mathrm{T}}$, $k \leqslant n$, where c_i are the coefficients of the linear expression of υ with respect to the eigenvectors x_i.

Step 5. For $c_j \neq 0$, $1 \leqslant j \leqslant k$, determine the vector $\xi_t = x_t - (\bar{c}_t/\bar{c}_j)x_j$, $t \neq j$, $1 \leqslant t \leqslant k$.

Step 6. Orthonormalize the vectors $\xi_1, \xi_2, \ldots, \xi_{k-1}$.

Step 7. The orthonormalized vectors define the matrix $P = [w_1 \;\; w_2 \;\; \cdots \;\; w_{k-1}]$.

Step 8. Calculate the matrix P^*AP and illustrate $NR[P^*AP]$.

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