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CANONICAL CORRELATIONS IN MULTI-WAY LAYOUT

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Abstract. In this paper, some formulas are proposed, which concern the numbers of unit canonical correlations in a multi-way layout. Different types of canonical correlations are considered and their connection with connectedness and orthogonality are examined.

Key words and phrases: Canonical correlation, projectors.

1. Introduction

In experimental design, it is usually necessary the comparison of treatments and it is well known that the multiple correlation is a measure of association between one variable and a set of other variables. In fact, it is shown to be the maximum correlation between one variable and a linear function of the others, which is also the same as the correlation between a variable and its minimum variance unbiased predictor in terms of the other variables. This concept was generalized by Hotelling (1936) to study the association between two sets of variables.

Over the past twenty years, Hotelling's canonical correlation analysis has received much attention. This may be due to the fact that canonical correlation analysis includes a number of multivariate techniques, including multiple regression analysis, canonical discriminant analysis, corresponding analysis, etc. In linear models, using the theory of generalized inverse of matrices, Khatri (1976) has shown that canonical correlation analysis can be extended to the case in which the covariance matrix of two sets of variables may be singular. Also, Yanai and Takane (1992) studied canonical correlation analysis subject to linear constraints.

The canonical correlation analysis is important because they include many multivariate statistical models as special cases. To explain the variations of k autonomous factors in a set of observations, we need to use a k-way layout. In matrix notation, the linear model may be written as

$$E(\boldsymbol{y}) = X_1\boldsymbol{b}_1 + X_2\boldsymbol{b}_2 + \cdots + X_k\boldsymbol{b}_k = X\boldsymbol{b}, \tag{1.1}$$

where $\boldsymbol{y}$ is a $n \times 1$ vector of all the observations, the vectors $\boldsymbol{b}_i$ $(i = 1, 2, \dots, k)$ consist of the effects (row, column, treatment, etc.), and X_i are $n \times n_i$ *design matrices* identifying the correspondence between the elements of $\boldsymbol{y}$ and $\boldsymbol{b}_i$ of the k-way layout. In this paper we consider the following types of canonical correlations:

(i) $\rho_h^{(1*2*\cdots*\ell)}$ between $X_1^T\boldsymbol{y}, X_2^T\boldsymbol{y}, \dots,$ and $X_\ell^T\boldsymbol{y}$, (*ℓ factors total*)

(ii) $\rho_h^{(1*2*\cdots*\ell|s)}$ between $X_1^TP_s\boldsymbol{y}, X_2^TP_s\boldsymbol{y}, \dots,$ and $X_\ell^TP_s\boldsymbol{y}$, with $s \neq 1, 2, \dots, \ell$ (*several factors each adjusted for the s-th factor*)

(iii) $\rho_h^{(1*2*\cdots*\ell\|s)}$ between $X_1^T\boldsymbol{y}, X_2^T\boldsymbol{y}, \ldots, X_{\ell-1}^T\boldsymbol{y}$ and $X_\ell^T P_s\boldsymbol{y}$, with $s \neq 1, 2, \ldots, \ell$ (*partial adjusted for the factor s*)
where in (ii) and (iii) the matrix P_s is a symmetric orthogonal projection onto the null space of X_s^T. The sign $*$ corresponds to the partition of $n \times (n_1 + \cdots + n_k)$ design matrix X in block submatrices, and $h = 1, \ldots, m$. The upper limit m of the index h denotes the number of positive canonical correlations and we write $h = 0$, when the associated vectors are uncorrelated.

Styan has studied the cases of canonical correlations ρ_h in two- and three-way layout. In Styan (1983, 1986), the inequalities $0 \leq \rho_m \leq \cdots \leq \rho_1 \leq 1$ are presented and the relationships among the numbers u, t, m has been referred, where u denotes the number of canonical correlations of unit value, t is the number of positive canonical correlations less than 1, and $m = u + t$ is equal to the number of positive canonical correlations. In Styan (1986), properties of the connectedness and orthogonality have been studied, when the three-way layout is considered

$$\text{completely connected} \Longleftrightarrow u_{1*2*3} = 2$$

and

$$\text{weakly orthogonal} \Longleftrightarrow t = 0.$$

This work is an extension the most of the results entitled "Canonical correlations in the three-way layout" by Styan (1986) given in the reference in the sense that we extend results from three-way layout to multi-way layout. Several equalities are presented for different types of the u's, (the numbers of the unit canonical correlations), and the concept of connectedness is generalized. A recurrence relation with a two-way layout is investigated, as well as the version of orthogonality.

2. Canonical correlations in a multi-way layout

Let $X = [X_1 \ \ldots \ X_k]$ be the design matrix for the k-way layout, where the submatrices X_i $(i = 1, \ldots, k)$, are of dimensions $n \times n_i$, with $n \geq n_1 + \cdots + n_k$. Without loss of generality we assume that $\operatorname{rank}(X_i) = \dim R(X_i) = n_i$, where $R(X_i)$ stands the range of X_i (i.e., the subspace spanned by the columns of X_i). Further, we remind that

$$H_i = X_i(X_i^T X_i)^{-1} X_i^T \tag{2.1}$$

is an $n \times n$ orthogonal projector onto $R(X_i)$, since $\det(X_i^T X_i) > 0$. A simplified form of this projector is

$$H_i = X_i X_i^- \qquad (i = 1, 2, \ldots, k)$$

where the superscript "$-$" is referred to a least squares generalized inverse, in the sense that $XX^-X = X$ and $X^-XX^- = X^-$, see p. 430, Lancaster and Tismenetsky (1984). If $\operatorname{rank}(X_i) < n_i$, clearly in (2.1) the $(X_i^T X_i)^{-1}$ has to be $(X_i^T X_i)^-$. With regard to the orthogonal projectors, note that if H_{12} is the orthogonal projector onto $R([X_1 \ \ X_2])$, then by Rao and Yanai (1979),

$$H_{12} = H_1 + H_2 \quad \text{if and only if} \quad X_1^T X_2 = \mathbf{0} \tag{2.2}$$

$$H_{12} = H_1 + H_2 - H_1 H_2 \quad \text{if and only if} \quad H_1 H_2 = H_2 H_1. \tag{2.3}$$

A generalization of (2.2) appears in the next statement.

LEMMA 2.1. *Let the matrix* $[X_1\ X_2\ \dots\ X_k]$ *is of full rank and* $H_{12\dots k}$ *be the orthogonal projector onto* $R([X_1\ X_2\ \dots\ X_k])$, *then:*

I. $H_{12\dots k} = H_1 + H_2 + \cdots + H_k \Longleftrightarrow X_i^T X_j = \mathbf{0},\ \forall i,j$

II. $H_{12\dots k} = H_i + H_{12\dots(i-1)(i+1)\dots k|i}$,

where $H_{12\dots(i-1)(i+1)\dots k|i}$ *is the orthogonal projection onto* $R(P_i[X_1\ \dots\ X_{i-1}\ X_{i+1}\ \dots\ X_k])$, *with* $P_i = I - H_i$ *and* H_i *as in* (2.1).

PROOF. By the successive relationships

$$H_{12\dots k} = H_1 + H_{23\dots k} \Longleftrightarrow X_1^T[X_2\ X_3\ \dots\ X_k] = \mathbf{0} \Longleftrightarrow X_1^T X_j = \mathbf{0}; \qquad j = 2,\dots,k$$

and

$$H_{23\dots k} = H_2 + H_{3\dots k} \Longleftrightarrow X_2^T[X_3\ \dots\ X_k] = \mathbf{0} \Longleftrightarrow X_2^T X_j = \mathbf{0}; \qquad j = 3,\dots,k$$

the first claim is proved easily. For II, having $P_i = I - H_i$, the relationship

$$X_i^T P_i X_j = X_i^T(I - X_i(X_i^T X_i)^{-1} X_i^T) X_j = \mathbf{0}; \qquad i \neq j$$

leads to $X_i^T P_i[X_1\ \dots\ X_{i-1}\ X_{i+1}\ \dots\ X_k] = \mathbf{0}$ and thus the second equation turns out to be true. □

In order to evaluate the whole k-way layout we define the number u of *unit canonical correlations* of type (i):

$$u_{1*2*\cdots*k} = \sum_{i=1}^{k} \operatorname{rank}(X_i) - \operatorname{rank}[X_1\ X_2\ \dots\ X_k] = \sum_{i=1}^{k} n_i - \operatorname{rank}[X_1\ X_2\ \dots\ X_k] = \dim N([X_1\ X_2\ \dots\ X_k]), \tag{2.4}$$

where $N([\cdot])$ stands the null space of matrix. For $k = 2$ (or $= 3$) the equation (2.4) is referred, Styan (1986), to u_{1*2} or u_{1*2*3}.

Since $\operatorname{rank}(X_i) = \operatorname{rank}(H_i)$, and

$$\begin{aligned}\operatorname{rank}(H_1 + \cdots + H_k) &= \dim\left\{\omega : \omega = \sum_{i=1}^{k} H_i y, y \in \mathbb{R}^n\right\} \\ &= \dim\left\{\omega : \omega = \sum_{i=1}^{k} X_i v_i; v_i = (X_i^T X_i)^{-1} X_i^T y\right\} \\ &= \dim\{\omega : \omega = [X_1\ X_2\ \dots\ X_k]v; v = [v_1^T\ v_2^T\ \dots\ v_k^T]^T\} \\ &= \operatorname{rank}[X_1\ X_2\ \dots\ X_k]\end{aligned}$$

we have

$$u_{1*2*\cdots*k} = \sum_{i=1}^{k} \operatorname{rank}(H_i) - \operatorname{rank}(H_1 + H_2 + \cdots + H_k). \tag{2.5}$$

For these numbers note the next properties.

PROPOSITION 2.1. *If the matrices* $[X_1 \ \dots \ X_s]$ *and* $[X_{s+1} \ \dots \ X_k]$ *have full rank, then*

$$u_{1*2*\dots*k} = u_{12\dots s*(s+1)\dots k} + u_{1*2*\dots*s} + u_{(s+1)*\dots*k}. \tag{2.6}$$

PROOF. By the case of two-way layout and the equation (2.4) we have:

$$\begin{aligned} u_{12\dots s*(s+1)\dots k} &= \text{rank}[X_1 \ \dots \ X_s] + \text{rank}[X_{s+1} \ \dots \ X_k] \\ &\quad - \text{rank}[X_1 \ \dots \ X_s \ X_{s+1} \ \dots \ X_k] \\ &= \sum_{i=1}^{s} \text{rank}(X_i) - u_{1*\dots*s} \\ &\quad + \sum_{i=s+1}^{k} \text{rank}(X_i) - u_{(s+1)*\dots*k} - \text{rank}[X_1 \ \dots \ X_k] \\ &= u_{1*\dots*k} - u_{1*\dots*s} - u_{(s+1)*\dots*k}. \end{aligned}$$

□

Clearly for $k = 2, 3$ by (2.6) we obtain the well known relationships as they have been presented by Styan (1986). Moreover, for $s = 1$

$$u_{1*2*\dots*k} = u_{1*23\dots k} + u_{2*3*\dots*k} \tag{2.7}$$

and generally

$$u_{1*2*\dots*k} = u_{s*12\dots(s-1)(s+1)\dots k} + u_{1*\dots*(s-1)*(s+1)*\dots*k}.$$

For the concept of connectedness, as it is presented in Styan (1986), by (2.7) we say:

COROLLARY 2.1. *Let the design matrix* $X = [X_1 \ \dots \ X_k]$ *for the* k*-way layout has full rank. A multi-way layout is completely connected for effects if and only if*

$$u_{1*2*\dots*k} = k - 1.$$

PROOF. The relationship (2.7) leads to

$$\begin{aligned} u_{1*2*\dots*k} &= u_{1*23\dots k} + u_{2*3*\dots*k} \\ &= u_{1*23\dots k} + u_{2*34\dots k} + u_{3*4*\dots*k} = \cdots \\ &= u_{1*23\dots k} + u_{2*34\dots k} + \cdots + u_{(k-1)*k}. \end{aligned} \tag{2.8}$$

Hence,

$$u_{1*2*\dots*k} = k - 1 \iff u_{i*(i+1)\dots k} = 1, \qquad \text{for} \quad i = 1, 2, \dots, k-1.$$

□

Therefore by (2.6), we obtain the connectedness *by parts*:

$$u_{1*2*\dots*k} = k - 1 \iff u_{12\dots s*(s+1)\dots k} = 1, \qquad u_{1*2*\dots*s} = s - 1, \qquad \text{and} \\ u_{(s+1)*\dots*k} = k - s - 1,$$

concluding that *the connectedness of total factors is equivalent to the connectedness of any partition of factors.* Moreover, by (2.7) we conclude that a k-way layout is *completely connected* if and only if for any partition of $X = [\widetilde{X}_1 \ \widetilde{X}_2]$ we have $\widetilde{u}_{1*2} = 1$. Hence, it is not possible some u's are zero, while others are greater than one, to make $u_{1*2*\cdots*k} = k - 1$.

Mentioning that in two-way layout

$$u_{1*2} = \dim(\mathcal{C}_1 \cap \mathcal{C}_2)$$

where $\mathcal{C}_i$ stands the column space of X_i, by (2.8) we have

$$u_{1*2*\cdots*k} = \sum_{i=1}^{k-1} \dim(\mathcal{C}_i \cap \mathcal{C}_{(i+1)\dots k}).$$

It is useful to notice the necessity that the columns of the design matrix X have to be linearly independent.

Example 1. Let $k = 3$ and $X_1 = I_2$, $X_2 = [1 \ 0]^T$, $X_3 = [0 \ 1]^T$. Then by (2.4) $u_{1*2*3} = 2 + 1 + 1 - 2 = 2 = k - 1$ as required, but $u_{1*23} = 2 + 2 - 2 = 2 \neq 1$ and $u_{2*3} = 1 + 1 - 2 = 0 \neq 1$. These results are coming out since the dimension of the subspace $span\{X_1, X_2, X_3\}$ is equal to 2 ($\neq 4$).

Also, involving the equation (2.5), we may express the number $u_{1\dots s*(s+1)\dots k}$ in terms of H_i as follows:

$$\begin{aligned} u_{12\dots s*(s+1)\dots k} &= \operatorname{rank}[X_1 \ \dots \ X_s] + \operatorname{rank}[X_{s+1} \ \dots \ X_k] - \operatorname{rank}[X_1 \ \dots \ X_k] \\ &= \operatorname{rank}\left[\sum_{i=1}^{s} H_i\right] + \operatorname{rank}\left[\sum_{i=s+1}^{k} H_i\right] - \operatorname{rank}\left[\sum_{i=1}^{k} H_i\right] \end{aligned}$$

and

$$\begin{aligned} u_{s*1\dots(s-1)(s+1)\dots k} &= \operatorname{rank}(X_s) + \operatorname{rank}[X_1 \ \dots \ X_{s-1} \ X_{s+1} \ \dots \ X_k] \\ &\quad - \operatorname{rank}[X_1 \ \dots \ X_k] \\ &= \operatorname{rank}(H_s) + \operatorname{rank}\left[\sum_{\substack{i=1 \\ i\neq s}}^{k} H_i\right] - \operatorname{rank}\left[\sum_{i=1}^{k} H_i\right]. \end{aligned}$$

For canonical correlations of type (ii) or (iii), we define as u the numbers:

$$(2.9) \quad \begin{cases} u_{1*2*\cdots*\ell|s} = \sum_{i=1}^{\ell} \operatorname{rank}(P_s X_i) - \operatorname{rank}(P_s[X_1 \ X_2 \ \dots \ X_\ell]) \\ u_{1*2*\cdots*\ell\|s} = \sum_{i=1}^{\ell-1} \operatorname{rank}(X_i) + \operatorname{rank}(P_s X_\ell) - \operatorname{rank}[X_1 \ X_2 \ \dots \ X_{\ell-1} \ P_s X_\ell] \end{cases}$$

where $s \neq 1, 2, \dots, \ell$ and $P_s = I - H_s = I - X_s(X_s^T X_s)^{-1} X_s^T$.

PROPOSITION 2.2. *The numbers $u_{1*2*\cdots*\ell|s}$, and $u_{1*23\ldots k|s}$ are related to the other numbers u of unit canonical correlations by the equations*

$$\text{I.}\quad u_{1*2*\cdots*\ell|s} = u_{1*2*\cdots*\ell*s} - \sum_{i=1}^{\ell} u_{s*i} \qquad (\ell \le k;\ s \ne 1,2,\ldots,\ell)$$

$$= u_{1*23\ldots\ell|s} + u_{2*3\ldots\ell|s} + \cdots + u_{(\ell-1)*\ell|s} \tag{2.10}$$

$$\text{II.}\quad u_{1*23\ldots k|s} = u_{s*1*23\ldots(s-1)(s+1)\ldots k} - u_{s*1} - u_{s*23\ldots(s-1)(s+1)\ldots k}$$

$$= u_{1*23\ldots k} - u_{s*1}, \qquad (s \in \{2,\ldots,k\}). \tag{2.11}$$

PROOF. By the relationships

$$R[X_s\ \ X_i] = R(X_s) \oplus R(P_s X_i)$$

and

$$\operatorname{rank}[X_s\ \ X_i] = \operatorname{rank}(X_s) + \operatorname{rank}(X_i) - u_{s*i}$$

clearly, $\operatorname{rank}(P_s X_i) = \operatorname{rank}(X_i) - u_{s*i}$.

I. The equation (2.4) and the previous statement lead to

$$\begin{aligned} u_{1*\cdots*\ell|s} &= \sum_{i=1}^{\ell} \operatorname{rank}(P_s X_i) - \operatorname{rank}(P_s[X_1\ \ldots\ X_\ell]) \\ &= \sum_{i=1}^{\ell} \operatorname{rank}[X_s\ X_i] - \ell \operatorname{rank}(X_s) - \operatorname{rank}[X_s\ X_1\ \ldots\ X_\ell] + \operatorname{rank}(X_s) \\ &= \sum_{i=1}^{\ell} \operatorname{rank}(X_i) + \operatorname{rank}(X_s) - \operatorname{rank}[X_s\ X_1\ \ldots\ X_\ell] - \sum_{i=1}^{\ell} u_{s*i} \\ &= u_{1*2*\cdots*\ell*s} - \sum_{i=1}^{\ell} u_{s*i}. \end{aligned}$$

Moreover, by (2.8) and the equation $u_{i*jk} = u_{i*k} + u_{i*j|k}$ of Theorem 2.1, Styan (1986), we take

$$\begin{aligned} u_{1*2*\cdots*\ell|s} &= u_{1*2*\cdots*\ell*s} - \sum_{i=1}^{\ell} u_{s*i} \\ &= u_{1*23\ldots\ell s} + u_{2*3\ldots\ell s} + \cdots + u_{(\ell-1)*\ell s} + u_{\ell*s} \\ &\quad - u_{s*1} - u_{s*2} - \cdots - u_{s*(\ell-1)} - u_{\ell*s} \\ &= u_{1*23\ldots\ell|s} + u_{2*3\ldots\ell|s} + \cdots + u_{(\ell-1)*\ell|s}. \end{aligned}$$

II. On the other hand we have:

$$\begin{aligned} u_{1*23\ldots k|s} &= \operatorname{rank}(P_s X_1) + \operatorname{rank}(P_s[X_2\ \ldots\ X_{s-1}\ X_{s+1}\ \ldots\ X_k]) \\ &\quad - \operatorname{rank}(P_s[X_1\ X_2\ \ldots\ X_{s-1}\ X_{s+1}\ \ldots\ X_k]) \\ &= \operatorname{rank}(X_1) - u_{s*1} + \operatorname{rank}[X_2\ \ldots\ X_{s-1}\ X_{s+1}\ \ldots\ X_k] - u_{s*2\ldots(s-1)(s+1)\ldots k} \\ &\quad - \operatorname{rank}[X_1\ X_2\ \ldots\ X_{s-1}\ X_{s+1}\ \ldots\ X_k] + u_{s*1\ldots(s-1)(s+1)\ldots k} \\ &= u_{1*23\ldots(s-1)(s+1)\ldots k} + u_{s*123\ldots(s-1)(s+1)\ldots k} - u_{s*1} - u_{s*2\ldots(s-1)(s+1)\ldots k} \\ &= u_{s*1*23\ldots(s-1)(s+1)\ldots k} - u_{s*1} - u_{s*23\ldots(s-1)(s+1)\ldots k} \end{aligned}$$

and using (2.7) we take:

$$\begin{aligned} u_{1*23\dots(s-1)(s+1)\dots k|s} &= \operatorname{rank}[X_s \;\; X_1] - \operatorname{rank}(X_s) + \operatorname{rank}[X_2 \;\dots\; X_k] \\ &\quad - \operatorname{rank}(X_s) - \operatorname{rank}[X_1 \;\dots\; X_k] + \operatorname{rank}(X_s) \\ &= \operatorname{rank}(X_1) - u_{s*1} + \sum_{i=2}^{k} \operatorname{rank}(X_i) - u_{2*\dots*k} - \operatorname{rank}[X_1 \;\dots\; X_k] \\ &= u_{1*\dots*k} - u_{2*\dots*k} - u_{s*1} = u_{1*23\dots k} - u_{s*1}. \end{aligned}$$ □

COROLLARY 2.2. *Supposing that* $s \neq 1, \dots, \ell$ *we have*

$$u_{1*\dots*\ell\|s} = u_{1*\dots*(\ell-1)} + u_{1\dots(\ell-1)*\ell\|s} \tag{2.12}$$

and if $s \in \{1, \dots, k\}$, *then*

$$u_{1*\dots*k\|s} = u_{1*\dots*(s-1)*(s+1)*\dots*(k-1)} + u_{12\dots(k-1)*k\|s}. \tag{2.13}$$

PROOF. Similarly, by (2.9) for (2.12) we have:

$$\begin{aligned} u_{1*\dots*\ell\|s} &= \sum_{i=1}^{\ell-1} \operatorname{rank}(X_i) + \operatorname{rank}(P_s X_\ell) - \operatorname{rank}[X_1 \;\dots\; X_{\ell-1} \;\; P_s X_\ell] \\ &= \sum_{i=1}^{\ell-1} \operatorname{rank}(X_i) - \operatorname{rank}[X_1 \;\dots\; X_{\ell-1}] + u_{12\dots(\ell-1)*\ell\|s} \\ &= u_{1*\dots*(\ell-1)} + u_{12\dots(\ell-1)*\ell\|s}. \end{aligned}$$

The equation (2.13) is a special case of (2.12). □

By the first equation of (2.10) and the equation (2.12), we say:

PROPOSITION 2.3. *A multi-way layout of the type* (ii) *or* (iii) *is connected for effects, i.e.,* $u_{1*\dots*\ell|s} = 0$ *if and only if*

$$u_{1*\dots*\ell*s} = \ell, \qquad \text{and} \qquad u_{s*i} = 1, \qquad i = 1, \dots, \ell, \; s \neq i,$$

while $u_{1*\dots*\ell\|s} = \ell - 2$ *if and only if*

$$u_{1*\dots*(\ell-1)} = \ell - 2, \qquad \text{and} \qquad u_{12\dots(\ell-1)*\ell\|s} = 0.$$

We define

$$m_{1*2*3*\dots*\ell} = \sum_{i=1}^{\ell-1} \operatorname{rank}(X_i^T [X_{i+1} \;\; X_{i+2} \;\dots\; X_\ell]) \tag{2.14}$$

and

$$t_{1*2*3*\dots*\ell} = \sum_{i=1}^{\ell-1} \operatorname{rank}(X_i^T P_{(i+1)\dots\ell} P_i [X_{i+1} \;\; X_{i+2} \;\dots\; X_\ell]), \tag{2.15}$$

where $P_{(i+1)\dots\ell} = I - H_{(i+1)\dots\ell}$. The above equations generalize the cases of m_{1*2} and t_{1*2} in a two-way layout form and declare that

$$m_{1*2*3*\cdots*(\ell-1)*\ell} = m_{1*23\dots\ell} + m_{2*3\dots\ell} + \cdots + m_{(\ell-1)*\ell} \tag{2.16}$$

and

$$t_{1*2*3*\cdots*(\ell-1)*\ell} = t_{1*23\dots\ell} + t_{2*3\dots\ell} + \cdots + t_{(\ell-1)*\ell}. \tag{2.17}$$

Similarly by (2.14) and (2.15), we define the numbers $m_{1*2*\cdots*\ell|s}$ and $t_{1*2*\cdots*\ell|s}$, as well as the numbers $m_{1*2*\cdots*\ell||s}$ and $t_{1*2*\cdots*\ell||s}$.

PROPOSITION 2.4. *The positive numbers m, u, t are interrelated by the equation*

$$m_{1*2*\cdots*k} = u_{1*2*\cdots*k} + t_{1*2*\cdots*k}. \tag{2.18}$$

PROOF. The equation (2.18) is well known for $k = 2$, Styan (1986), and it is based on the equation, Bérubé *et al.* (1993),

$$\operatorname{rank}(M^T N) = \operatorname{rank}(M) + \operatorname{rank}(N) - \operatorname{rank}([M \;\; N]) + \operatorname{rank}(M^T P_N P_M N).$$

Using this relationship (i.e., for two factors) we have

$$\begin{aligned} m_{1*23\dots k} &= u_{1*23\dots k} + t_{1*23\dots k}, \\ m_{2*3\dots k} &= u_{2*3\dots k} + t_{2*3\dots k}, \\ &\vdots \\ m_{(k-1)*k} &= u_{(k-1)*k} + t_{(k-1)*k}. \end{aligned}$$

Thus, adding these equations, by (2.8), (2.16) and (2.17), the equation (2.18) follows immediately. □

Note that (2.18) can be proved also by induction.

COROLLARY 2.3.

$$\begin{cases} m_{1*2*\cdots*(\ell-1)*\ell|s} = u_{1*2*\cdots*(\ell-1)*\ell|s} + t_{1*2*\cdots*(\ell-1)*\ell|s} \\ m_{1*2*\cdots*(\ell-1)*\ell||s} = u_{1*2*\cdots*(\ell-1)*\ell||s} + t_{1*2*\cdots*(\ell-1)*\ell||s}. \end{cases} \tag{2.19}$$

PROOF. Using the equation in two-way layout, Styan (1983, 1986),

$$m_{i*j|k} = u_{i*j|k} + t_{i*j|k},$$

the second of (2.10) and the equations

$$\begin{aligned} m_{1*2*\cdots*(\ell-1)*\ell|s} &= \sum_{i=1}^{\ell-1} \operatorname{rank}(X_i^T P_s [X_{i+1} \;\; X_{i+2} \;\; \dots \;\; X_\ell]) \\ &= \sum_{i=1}^{\ell-1} m_{i*(i+1)(i+2)\dots\ell|s} \end{aligned}$$

$$t_{1*2*\cdots*(\ell-1)*\ell|s} = \sum_{i=1}^{\ell-1} \operatorname{rank}(X_i^T P_{(i+1)\ldots\ell s} P_{is} [X_{i+1} \ \ X_{i+2} \ \ \ldots \ \ X_\ell]) = \sum_{i=1}^{\ell-1} t_{i*(i+1)(i+2)\ldots\ell|s},$$

the first of (2.19) is fulfilled. The same for the second of (2.19). □

Analogous generalizations of the condition in Section 3 on the two-way layout, Baksalary *et al.* (1992), which characterize the situation $m = t$ (i.e. when there are not unit canonical correlations), we derive also by (2.16) and (2.17).

It is worth noting that the orthogonality $X_i^T X_s = \mathbf{0}$ $(i = 1, \ldots, \ell)$, due to $P_s X_i = X_i$ and $P_{si} = I - H_{si} = I - H_s - H_i = P_s P_i = P_i P_s$, leads to the identities

$$m_{1*2*\cdots*\ell|s} = \sum_{i=1}^{\ell-1} \operatorname{rank}(X_i^T P_s [X_{i+1} \ \ \ldots \ \ X_\ell]) = m_{1*2*\cdots*\ell}$$

and

$$\begin{aligned} t_{1*2*\cdots*\ell|s} &= \sum_{i=1}^{\ell-1} \operatorname{rank}(X_i^T P_{(i+1)\ldots\ell s} P_{is} [X_{i+1} \ \ \ldots \ \ X_\ell]) \\ &= \sum_{i=1}^{\ell-1} \operatorname{rank}(X_i^T P_s P_{(i+1)\ldots\ell} P_i P_s [X_{i+1} \ \ \ldots \ \ X_\ell]) \\ &= \sum_{i=1}^{\ell-1} \operatorname{rank}(X_i^T P_{(i+1)\ldots\ell} P_i [X_{i+1} \ \ \ldots \ \ X_\ell]) = t_{1*2*\cdots*\ell}, \end{aligned}$$

consequently

$$u_{1*2*\cdots*\ell|s} = u_{1*2*\cdots*\ell}.$$

Similarly for the numbers m, u, t of type (iii).

3. Results on orthogonality

Using the notation of canonical correlations in the introduction, we generalize in the following, some results on weak orthogonality of rows and columns as the $\boldsymbol{y}$ stated in two-way layout in Styan (1986) and Bérubé *et al.* (1993). The next proposition refers to the relationships of orthogonality and connectedness.

PROPOSITION 3.1. *If $s \notin \{1, \ldots, \ell\}$, then $m_{1*2*\cdots*\ell|s} = 0$ and $u_{i*s} = 1$ if and only if $t_{1*2*\cdots*\ell|s} = 0$ and $u_{i*(i+1)\ldots\ell s} = 1$, for $i = 1, \ldots, \ell - 1$.*

PROOF. Clearly, $m_{1*2*\cdots*\ell|s} = 0 \iff u_{1*2*\cdots*\ell|s} = t_{1*2*\cdots*\ell|s} = 0$. By Proposition 2.2, the equation $u_{1*2*\cdots*\ell|s} = 0$ implies $u_{i*(i+1)\ldots\ell|s} = 0$ for $i = 1, \ldots, \ell - 1$ and then $u_{i*(i+1)\ldots\ell s} - u_{i*s} = 0$, i.e., $u_{i*(i+1)\ldots\ell s} = 1$.

Inversely, by $t_{1*2*\cdots*\ell|s} = 0$ and the first of (2.19), we take the relationship $m_{1*2*\cdots*\ell|s} = u_{1*2*\cdots*\ell|s}$. If $u_{1*2*\cdots*\ell|s} > 0$, then by (2.10) we have $\sum_{i=1}^{\ell-1} u_{i*(i+1)\ldots\ell|s} > 0$. In this sum there exist some positive terms, let $u_{i*(i+1)\ldots\ell|s}$. Since, $u_{i*(i+1)\ldots\ell|s} =$

$u_{i*(i+1)\dots\ell s} - u_{i*s} > 0 \Rightarrow u_{i*s} < 1$, absurd. Hence, $u_{1*2*\dots*\ell|s} = 0$, i.e., $m_{1*2*\dots*\ell|s} = 0$, and $u_{i*s} = 1$, for $i = 1, \dots, \ell - 1$. □

Denoting by $ch_h(\cdot)$ the h-th largest real eigenvalue, in two-way layout, it is known, Kharti (1976) and Bérubé *et al.* (1993), that the canonical correlation

$$\rho_h^{(i*j)} = ch_h^{1/2}(H_i H_j).$$

In multi-way layout the canonical correlation of type (i) and (ii) are generalized respectively as follows

$$\rho_h^{(1*2*\dots*\ell)} = ch_h^{1/2}(H_1 H_2 \cdots H_\ell)$$
$$\rho_h^{(1*2*\dots*\ell|s)} = ch_h^{1/2}(H_{1|s} H_{2|s} \cdots H_{\ell|s}).$$

A relationship between ρ_h of type (i) and (ii) is the next statement.

PROPOSITION 3.2. *Let the design matrices $X_1, \dots, X_\ell, X_s$ be weakly orthogonal, i.e., $t_{i*j} = 0$ for $i, j = 1, 2, \dots, \ell-1$, $j > i$ and $t_{s*i} = 0$, $i = 1, 2, \dots, \ell$, then the algebraic multiplicity of eigenvalue $\lambda = 1$ of matrix $H_1 H_2 \cdots H_{\ell-1} H_s$ is equal to*

$$\{\rho_h^{(1*2*\dots*(\ell-1)*\ell s)}\} - \{\rho_h^{(1*2*\dots*(\ell-1)*\ell|s)}\},$$

where $\{\rho_h^{(\cdot)}\}$ denotes the set of canonical correlations of factors.

PROOF. We follow analogue statements of Theorem 3 in Bérubé *et al.* (1993). By the equation II in Lemma 2.1, we have

$$\begin{aligned} &|\lambda I - H_1 H_2 \cdots H_{\ell-1} H_{\ell s}| \\ &= |\lambda I - H_1 H_2 \cdots H_{\ell-1}(H_s + H_{\ell|s})| \\ &= |\lambda I - H_1 \cdots H_{\ell-1} H_s| \\ &\quad \times |I - (\lambda I - H_1 H_2 \cdots H_{\ell-1} H_s)^{-1} H_1 H_2 \cdots H_{\ell-1} H_{\ell|s}| \end{aligned} \tag{3.1}$$

where λ is not eigenvalue of $H_1 \cdots H_{\ell-1} H_s$. Since $t_{i*j} = 0$ the matrices H_i, H_j are commuting, Bérubé *et al.* (1993), and the matrix $H_1 H_2 \cdots H_{\ell-1} H_s$ is idempotent. Moreover, by the equation (2.3) and II in Lemma 2.1 we have

$$H_{j|i} = H_j - H_i H_j.$$

Thus

$$(\lambda I - H_1 \cdots H_{\ell-1} H_s) H_1 H_2 \cdots H_{\ell-1} H_{\ell|s} = \lambda H_1 H_2 \cdots H_{\ell-1} H_{\ell|s}$$

due to $H_s H_{\ell|s} = H_s(H_\ell - H_s H_\ell) = \mathbf{0}$. Therefore the equation (3.1) is equivalent to

$$\begin{aligned} |\lambda I - H_1 H_2 \cdots H_{\ell-1} H_{\ell s}| &= |\lambda I - H_1 H_2 \cdots H_{\ell-1} H_s| |I - \lambda^{-1} H_1 H_2 \cdots H_{\ell-1} H_{\ell|s}| \\ &= \left(\frac{\lambda - 1}{\lambda}\right)^q |\lambda I - H_1 H_2 \cdots H_{\ell-1} H_{\ell|s}| \end{aligned} \tag{3.2}$$

where $q = u_{1*2*\cdots*(\ell-1)*s}$. On the other hand

$$\begin{aligned} H_1H_2\cdots H_{\ell-1}H_{\ell|s} &= H_1H_2\cdots H_{\ell-1}H_{\ell|s} - H_1\cdots H_{\ell-1}H_sH_{\ell|s} \\ &= H_1H_2\cdots H_{\ell-2}(H_{\ell-1} - H_{\ell-1}H_s)H_{\ell|s} \\ &= H_1\cdots H_{\ell-2}H_{\ell-1|s}H_{\ell|s} \\ &\vdots \\ &= H_{1|s}H_{2|s}\cdots H_{\ell-1|s}H_{\ell|s}. \end{aligned}$$

Hence, we obtain the required result, by

$$|\lambda I - H_1\cdots H_{\ell-1}H_{\ell s}| = \left(\frac{\lambda-1}{\lambda}\right)^q |\lambda I - H_{1|s}\cdots H_{\ell|s}|. \qquad \square$$

Now, if one factor, say k, is pairwise orthogonal to all of the other factors, the next relationship arises.

COROLLARY 3.1. *Let $k \in \{1,\dots,\ell-1\}$ and let $t_{s*i} = 0$, for $i = 1,2,\dots,\ell-1,\ell$, and $t_{i*j} = 0$, for $i,j = 1,2,\dots,\ell-1,\ell$, $j > i$, with $i,j \neq k$, then the difference*

$$\{\rho_h^{(1*\cdots*(k-1)*(k+1)*\cdots*\ell*ks)}\} - \{\rho_h^{(1*\cdots*(k-1)*(k+1)*\cdots*\ell*k|s)}\}$$

is equal to the algebraic multiplicity of eigenvalue $\lambda = 1$ of matrix

$$H_1\cdots H_{k-1}H_{k+1}\cdots H_\ell H_s.$$

Moreover, at once from (3.2) the following result is obtained:

COROLLARY 3.2. *Let $t_{s*j} = 0$, for $j = 1,\dots,\ell$, and $t_{i*j} = 0$, with $j > i$, where $i,j = 1,\dots,\ell$, then,*

$$\{\rho_h^{(1*2*\cdots*(\ell-1)*\ell s)}\} - \{\rho_h^{(1*2*\cdots*(\ell-2)*\ell*(\ell-1)s}\} = u_{1*2*\cdots*(\ell-1)*s} - u_{1*2*\cdots*(\ell-2)*\ell*s}$$

is equal to the difference of algebraic multiplicities of eigenvalue $\lambda = 1$ of matrices $H_1H_2\cdots H_{\ell-1}H_s$ and $H_1H_2\cdots H_{(\ell-2)}H_\ell H_s$.

The proof is similar of Theorem 3 in Bérubé *et al.* (1993).

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