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On numerical ranges of the compressions of normal matrices

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ABSTRACT

For an $n \times n$ normal matrix A, whose numerical range $NR[A]$ is a k-polygon ($k \leqslant n$), an $n \times (k-1)$ isometry matrix P is constructed by a unit vector $\upsilon \in \mathbb{C}^n$, and $NR[P^*AP]$ is inscribed to $NR[A]$. In this paper, using the notations of $NR[P^*AP]$ and some properties from projective geometry, an $n \times n$ diagonal matrix B and an $n \times (k-2)$ isometry matrix Q are proposed such that $NR[P^*AP]$ and $NR[Q^*BQ]$ have as common support lines the edges of the k-polygon and share the same boundary points with the polygon. It is proved that the boundary of $NR[P^*AP]$ is a differentiable curve and the boundary of the numerical range of a 3×3 matrix P^*AP is an ellipse, when the polygon is a quadrilateral.

1. Notation and preliminaries

We begin by settling on the notation to be used. Let $\mathcal{M}_n$ denote the algebra of all $n \times n$ complex matrices, and let $A \in \mathcal{M}_n$. The *numerical range* of A, also known as the *field of values* [11], is the set

$$NR[A] = \{x^*Ax \in \mathbb{C} : x \in \mathbb{C}^n \text{ with } x^*x = 1\}.$$

For a review of this topic see [13] and the well-known books [10,11] on the numerical range and matrix analysis, especially the applications, that are related to the numerical solution of a partial differential equation or the stability of an equilibrium in a dynamical system governed by a system of differential equations. Let us briefly recall the basic properties of $NR[A]$, (see [11, Chapter 1]). $NR[A]$ is a nonempty compact and convex subset of $\mathbb{C}$, that contains the spectrum $\sigma(A)$ of A. When A is a real matrix, then $NR[A]$ is symmetric with respect to the real axis. The numerical range of a 2×2 matrix A is a closed elliptical disk, whose boundary is an ellipse with foci at its eigenvalues λ_1, λ_2 of A, and center at $trA/2$; the length of major axis is equal to $(tr(A^*A) + 2|detA|)^{1/2}$ and of the minor axis $(tr(A^*A) - |\lambda_1|^2 - |\lambda_2|^2)^{1/2}$, [11, p. 23]. When A is expressed as direct sum of matrices $A = A_1 \oplus A_2 \oplus \cdots \oplus A_\ell$, then $NR[A] = \mathrm{Co}\{NR[A_1] \cup NR[A_2] \cup \cdots \cup NR[A_\ell]\}$, where $\mathrm{Co}\{X\}$ denotes the convex hull of a set X. We write $A = H_A + \mathbf{i}S_A$, where

$$H_A = \frac{A + A^*}{2} \quad \text{and} \quad S_A = \frac{A - A^*}{2\mathbf{i}}$$

are the Hermitian matrices with the real and imaginary part of A, respectively. A is Hermitian matrix if and only if $NR[A] \subset \mathbb{R}$, and if A is normal, then $NR[A] = \mathrm{Co}\{\sigma(A)\}$. Also, a point $\lambda \in \sigma(A)$ is a *normal eigenvalue* of A, if it belongs to the boundary $\partial NR[A]$ of the numerical range; namely, there exists a unitary matrix $U \in \mathcal{M}_n$ such that A is unitarily similar to $U^*AU = \lambda I_m \oplus B$, where m is the algebraic multiplicity of λ and $\lambda \notin \sigma(B)$, [11, Theorem 1.6.6]. The numerical range is invariant under unitary similarity; that is, $NR[A] = NR[U^*AU]$, where $U \in \mathcal{M}_n$ is a unitary matrix.

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Given two matrices $A \in \mathcal{M}_n$ and $C \in \mathcal{M}_k$,with $1 \leqslant k < n$, the matrix C is said to be a $k-$ *compression* of A, if there exists an $n \times k$ orthonormal matrix P satisfying

$$P^*P = I_k \quad \text{and} \quad C = P^*AP.$$

Sometimes in literature, the matrix P is called an *isometry* and $P^*A\,P$ is called an *isometric projection* of A. For the numerical range of a k-compression of A holds

$$NR[C] = NR[P^*AP] \subseteq NR[A]. \tag{1}$$

The numerical range of compressions of normal matrices have attracted attention and several results have been published in [1–4,8,9]. The inclusion relation in (1) has been presented in details in [1,2], where the investigation leads to a structure of P such that the boundaries of the numerical ranges $NR[P^*AP]$ and $NR[A]$ have common points.

To explain, let a normal matrix $A \in \mathcal{M}_n$ and the convex k-polygon

$$\mathcal{P} = \langle \lambda_1, \lambda_2, \ldots, \lambda_k \rangle = \operatorname{Co}\{\sigma(A)\} = NR[A],$$

where the eigenvalues λ_i, $i = 1,2,\ldots,k$, are distinct corresponding to the vertices of $\mathcal{P}$, and let $\{x_1, x_2, \ldots, x_k\}$ be an orthonormal system of eigenvectors of A corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively and $W_x = \operatorname{span}\{x_1, x_2, \ldots, x_k\}$. For every unit vector

$$\upsilon = \sum_{i=1}^{k} \upsilon_i x_i \in W_x; \quad \upsilon_i \in \mathbb{C} \setminus \{0\}, \quad i = 1,2,\ldots,k, \tag{2}$$

the point $\upsilon^*A\upsilon$ lies inside the polygon $\mathcal{P}$. In the following, the inner product between two vectors is denoted by $\circ$ and the orthogonal complement of span$\{\upsilon\}$ with respect to the subspace W_x by $E^{\perp}_{W_x}(\upsilon)$. Since for the vector $g = g_1x_1 + \cdots + g_kx_k \in E^{\perp}_{W_x}(\upsilon)$ holds $g \circ \upsilon = \sum_{i=1}^{k} \bar{\upsilon}_i g_i = 0$, for any index $1 \leqslant j \leqslant k$, we may write

$$g = g_1\left(x_1 - \frac{\bar{\upsilon}_1}{\bar{\upsilon}_j}x_j\right) + g_2\left(x_2 - \frac{\bar{\upsilon}_2}{\bar{\upsilon}_j}x_j\right) + \cdots + g_k\left(x_k - \frac{\bar{\upsilon}_k}{\bar{\upsilon}_j}x_j\right).$$

Therefore, orthonormalizing the vectors

$$b_1 = x_1 - \frac{\bar{\upsilon}_1}{\bar{\upsilon}_j}x_j, \ldots, b_{j-1} = x_{j-1} - \frac{\bar{\upsilon}_{j-1}}{\bar{\upsilon}_j}x_j, \quad b_j = x_{j+1} - \frac{\bar{\upsilon}_{j+1}}{\bar{\upsilon}_j}x_j, \ldots, b_{k-1} = x_k - \frac{\bar{\upsilon}_k}{\bar{\upsilon}_j}x_j,$$

we may construct a basis $\{w_1, w_2, \ldots, w_{k-1}\}$ of $E^{\perp}_{W_x}(\upsilon)$, which in turn defines the isometry $n \times (k-1)$ matrix

$$P = [\,w_1 \quad w_2 \quad \cdots \quad w_{k-1}\,], \tag{3}$$

and the corresponding $(k-1)$-compression of A, $C = P^*A\,P$. It is evident that

$$\begin{aligned} NR[C] &= \left\{(P\psi)^*A(P\psi) : \psi \in \mathbb{C}^{k-1}, \psi^*\psi = 1\right\} = \left\{x^*Ax : x \in E^{\perp}_{W_x}(\upsilon), x^*x = 1\right\} \\ &\subseteq \{x^*Ax : x \in W_x, x^*x = 1\} = \langle \lambda_1, \lambda_2, \ldots, \lambda_k \rangle \equiv \mathcal{P}. \end{aligned}$$

Furthermore, considering another basis of $E^{\perp}_{W_x}(\upsilon)$, consisted of the unit vectors

$$y_i = \frac{\bar{\upsilon}_{i+1}}{\sqrt{|\upsilon_i|^2 + |\upsilon_{i+1}|^2}}x_i - \frac{\bar{\upsilon}_i}{\sqrt{|\upsilon_i|^2 + |\upsilon_{i+1}|^2}}x_{i+1} \in \mathbb{C}^n; \quad i = 1,2,\ldots,k \tag{4}$$

with $x_{k+1} = x_1$ and $\upsilon_{k+1} = \upsilon_1$, the points

$$c_i \equiv c_i(\upsilon) = y_i^*Ay_i = \frac{|\upsilon_{i+1}|^2\lambda_i + |\upsilon_i|^2\lambda_{i+1}}{|\upsilon_i|^2 + |\upsilon_{i+1}|^2}; \quad i = 1,2,\ldots,k; \quad \lambda_{k+1} = \lambda_1 \tag{5}$$

are defined. Note that each c_i depends on the components of the unit vector υ, and that c_i is an interior point of the closed line segment $\langle \lambda_i, \lambda_{i+1} \rangle$, $i = 1,2, \ldots, k$. Indeed, by (5) $c_i \in \operatorname{Co}\{\lambda_i, \lambda_{i+1}\} = \langle \lambda_i, \lambda_{i+1} \rangle \subset \partial NR[A]$, and since $\upsilon_i \neq 0$ (see (2)), it is obvious that $c_i \neq \lambda_i, \lambda_{i+1}$. Thus for $t \in (0,1)$, we may write $c_i = t\lambda_i + (1-t)\lambda_{i+1}$, which leads to

$$\frac{c_i - \lambda_{i+1}}{\lambda_i - c_i} = \frac{t}{1-t} \in \mathbb{R}^+; \quad i = 1,2,\ldots,k. \tag{6}$$

Furthermore [1, Theorem 1] yields $\partial NR[A] \cap \partial NR[C] = \{c_1, \ldots, c_k\}$.

In the next section, we consider an $n \times n$ normal matrix $A \in \mathcal{M}_n$, $n \geqslant 2$, with $NR[A]$ a k-polygon $\mathcal{P}$, whose eigenvalues and an interior point of $NR[A]$ define a diagonal $n \times n$ matrix B, with $NR[B]$ a $(k-1)$-polygon $\mathcal{Q}$. Given a $(k-1)$-compression P^*AP of matrix A, we present necessary conditions for the existence of a unit vector, for which the corresponding $(k-2)$-compression Q^*BQ of B is such that $\partial NR[P^*AP]$ and $\partial NR[Q^*BQ]$ are inscribed to the same polygon $\mathcal{P}$ at the common (boundary) points c_i in (5). The inverse problem is discussed in Theorem 4. In the last section of this article, some properties of the boundary curve of the numerical range of the compression are investigated and some applications are presented. In particular, in Proposition 6 additional conditions are formulated so that the sets, $\partial NR[P^*AP]$, $\partial NR[Q^*BQ]$ and an ellipse, share the same points.

Also, it is proved that the boundary of the numerical range of a compression is a differentiable curve, with no sharp points and line segments (Theorem 8). As a consequence of these properties it is proved that the numerical range of a 3×3 matrix P^*AP is an elliptical disk, when the numerical range of A is a quadrilateral.

2. General results

Consider a normal matrix $A \in \mathcal{M}_n$, and $\mathcal{P}$ the convex polygon, with vertices λ_i, $i = 1,2,..,k$, the simple eigenvalues of A, all the other eigenvalues of A being interior points of $NR[A]$. Denoting by

$$B = diag(\lambda_1, \lambda_2, \dots, \lambda_{k-2}, \lambda_s, \mu, \lambda_{k+1}, \dots, \lambda_n) \in \mathcal{M}_n, \tag{7}$$

where $\lambda_s = \langle \lambda_{k-2}, \lambda_{k-1} \rangle \cap \langle \lambda_k, \lambda_1 \rangle$ and μ is an interior point of $NR[A]$, then

$$NR[B] = \mathrm{Co}\{\sigma(B)\} = \langle \lambda_1, \lambda_2, \dots, \lambda_{k-2}, \lambda_s \rangle.$$

If $W_e = \mathrm{span}\,\{e_1, e_2, \dots, e_{k-1}\}$,where e_i are the vectors of the standard basis of $\mathbb{C}^n$, then clearly for every unit vector

$$z = \sum_{i=1}^{k-1} z_i e_i \in W_e; \quad z_i \in \mathbb{C} \setminus \{0\}, \quad i = 1, 2, \dots, k-1, \tag{8}$$

there exists an $n \times (k-2)$ isometry matrix Q, constructed as P in (3), such that the numerical range of the compression Q^*BQ is inscribed to the convex $(k-1)$-polygon

$$\mathcal{Q} = \langle \lambda_1, \lambda_2, \dots, \lambda_{k-2}, \lambda_s \rangle = NR[B].$$

The following theorem gives formulae for the components z_j, $j = 1,2,\dots,k-1$, of the unit vector z in (8), when a normal matrix $A \in \mathcal{M}_n$ and a unit vector $\upsilon \in \mathbb{C}^n$ as in (2) are given.

Theorem 1. *Let $A \in \mathcal{M}_n$ be a normal matrix, whose numerical range is a k-polygon $\mathcal{P}$. Assume that for a unit vector $\upsilon \in \mathbb{C}^n$ a $(k-1)$-compression P^*AP is defined and for the diagonal matrix $B \in \mathcal{M}_n$ in (7), there exists a unit vector $z \in \mathbb{C}^n$ as in (8), for which the corresponding isometry matrix Q and a $(k-2)$-compression Q^*BQ are constructed, such that*

(i) $\partial NR[P^*AP] \cap \langle \lambda_i, \lambda_{i+1} \rangle = \partial NR[Q^*BQ] \cap \langle \lambda_i, \lambda_{i+1} \rangle = \{c_i\}$; $i = 1,2,\dots,k-2,k$,
(ii) the ratios $\frac{c_{k-2}-\lambda_s}{\lambda_{k-2}-c_{k-2}}$, $\frac{c_k-\lambda_s}{\lambda_1-c_k}$ are positive numbers, and
(iii) $\frac{c_k-\lambda_s}{\lambda_1-c_k} = \frac{c_{k-2}-\lambda_s}{\lambda_{k-2}-c_{k-2}} \prod_{j=1}^{k-3} \frac{c_j-\lambda_{j+1}}{\lambda_j-c_j}$,

where c_i, $i = 1,2,\dots,k-2,k$, are given by (5). Then the components z_j, $j = 1,2,\dots,k-1$, of the unit vector z are defined by the equations

$$|z_{i+1}|^2 = |z_1|^2 \prod_{j=1}^{i} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j}; \quad i = 1, 2, \dots, k-3, \quad \textit{and} \quad |z_{k-1}|^2 = |z_1|^2 \frac{c_k - \lambda_s}{\lambda_1 - c_k} \tag{9}$$

with

$$|z_1|^2 = \left(1 + \frac{c_1 - \lambda_2}{\lambda_1 - c_1} + \prod_{j=1}^{2} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j} + \cdots + \prod_{j=1}^{k-3} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j} + \frac{c_k - \lambda_s}{\lambda_1 - c_k} \right)^{-1}. \tag{10}$$

Proof. From the unit vector $\upsilon \in \mathbb{C}^n$ an isometry $n \times (k-1)$ matrix P is constructed as in (3) and by [1, Theorem 1] a $(k-1)$-compression P^*AP is defined, for which holds $\partial NR[P^*AP] \cap \mathcal{P} = \{c_1, c_2, \dots, c_k\}$, where c_i depends on the unit vector υ and is given by (5), i.e.,

$$c_i \equiv c_i(\upsilon) = \frac{|\upsilon_{i+1}|^2 \lambda_i + |\upsilon_i|^2 \lambda_{i+1}}{|\upsilon_i|^2 + |\upsilon_{i+1}|^2}; \quad i = 1, 2, \dots, k$$

with $\lambda_{k+1} = \lambda_1$ and $\upsilon_{k+1} = \upsilon_1$. Applying the procedure mentioned in Section 1 to the diagonal matrix B in (7), using a unit vector $z = z_1 e_1 + z_2 e_2 + \ldots + z_{k-1} e_{k-1} \in \mathbb{C}^n$, when it exists, a $(k-2)-$ compression Q^*BQ is constructed and $\partial\, NR[Q^*BQ]$ is inscribed in the polygon $\mathcal{Q} = \langle \lambda_1, \lambda_2, \dots, \lambda_{k-2}, \lambda_s \rangle$ at the points

$$c_i \equiv c_i(z) = \frac{|z_{i+1}|^2 \lambda_i + |z_i|^2 \lambda_{i+1}}{|z_i|^2 + |z_{i+1}|^2}; \quad i = 1, 2, \dots, k-3, \tag{11}$$

$$c_{k-2} \equiv c_{k-2}(z) = \frac{|z_{k-1}|^2 \lambda_{k-2} + |z_{k-2}|^2 \lambda_s}{|z_{k-1}|^2 + |z_{k-2}|^2}, \tag{12}$$

$$c_k \equiv c_k(z) = \frac{|z_1|^2\lambda_s + |z_{k-1}|^2\lambda_1}{|z_1|^2 + |z_{k-1}|^2}. \tag{13}$$

For every $i = 1,2,\ldots,k-3$, from the expressions (5) and (11) of the common boundary points $c_i \in \partial NR[A] \cap \partial NR[B] \cap \partial NR[P^*AP] \cap \partial NR[Q^*BQ]$, we have

$$\frac{|\upsilon_{i+1}|^2}{|\upsilon_i|^2} = \frac{|z_{i+1}|^2}{|z_i|^2} = \frac{c_i - \lambda_{i+1}}{\lambda_i - c_i}, \tag{14}$$

noting here that c_i is the simplified formulation of $c_i(\upsilon)$ and $c_i(z)$. Since every ratio in (14) is well defined by (6), we can derive

$$\frac{|z_{i+1}|^2}{|z_1|^2} = \prod_{j=1}^{i} \frac{|z_{j+1}|^2}{|z_j|^2} = \prod_{j=1}^{i} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j}; \quad i = 1,2,\ldots,k-3,$$

from which the first formula of (9) is immediately deduced.

Moreover, since $c_{k-2} \in \partial NR[A] \cap \partial NR[B] \cap \partial NR[P^*AP] \cap \partial NR[Q^*BQ]$, thus by (5) and (12) we have

$$\frac{|\upsilon_{k-1}|^2}{|\upsilon_{k-2}|^2} = \frac{c_{k-2} - \lambda_{k-1}}{\lambda_{k-2} - c_{k-2}}, \quad \frac{|z_{k-1}|^2}{|z_{k-2}|^2} = \frac{c_{k-2} - \lambda_s}{\lambda_{k-2} - c_{k-2}} \tag{15}$$

and due to $c_k \in \partial NR[A] \cap \partial NR[B] \cap \partial\ NR[P^*AP] \cap \partial NR[Q^*BQ]$ by (5) and (13) we derive

$$\frac{|\upsilon_k|^2}{|\upsilon_1|^2} = \frac{c_k - \lambda_k}{\lambda_1 - c_k}, \quad \frac{|z_{k-1}|^2}{|z_1|^2} = \frac{c_k - \lambda_s}{\lambda_1 - c_k}, \tag{16}$$

whereby the second formula in (9) immediately arises. Note that, when the points $c_{k-2} \equiv c_{k-2}(\upsilon)$, $c_k \equiv c_k(\upsilon)$ are given by (5), the ratio of the first formula in (15) (respectively in (16)) is well defined by (6) and the ratio of the second formula in (15) (respectively in (16)) is well defined by assumption (ii). Also, we have to note that the second formula in (16) is proved substituting in the second formula of (15), the first formula in (9) for $i = k-3$, and using assumption (iii), i.e.,

$$|z_{k-1}|^2 = |z_{k-2}|^2 \frac{c_{k-2} - \lambda_s}{\lambda_{k-2} - c_{k-2}} = |z_1|^2 \frac{c_{k-2} - \lambda_s}{\lambda_{k-2} - c_{k-2}} \prod_{j=1}^{k-3} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j} = |z_1|^2 \frac{c_k - \lambda_s}{\lambda_1 - c_k}.$$

Furthermore, since the vector $z \in \mathbb{C}^n$ has to be a unit vector, combining $\sum_{i=1}^{k-1} |z_i|^2 = 1$ and (9) we conclude

$$|z_1|^2 \left(1 + \frac{c_1 - \lambda_2}{\lambda_1 - c_1} + \prod_{j=1}^{2} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j} + \cdots + \prod_{j=1}^{k-3} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j} + \frac{c_k - \lambda_s}{\lambda_1 - c_k} \right) = 1,$$

whereby the expression of z_1 in (10) is obvious. □

Remark 2

1. We have to note that the uniqueness of the formula of the components z_j, $j = 1,2,\ldots,k-1$, requires assumption (iii) in Theorem 1. Indeed, since

   $$\frac{|z_{k-1}|^2}{|z_1|^2} = \frac{|z_{k-1}|^2}{|z_{k-2}|^2} \frac{|z_{k-2}|^2}{|z_1|^2},$$

   the substitution of all the above ratios from the second formulae of (16), (15) and the first formula of (9) for $i = k-3$ yields the equality in assumption (iii), which is now formulated as

   $$\frac{(c_k - \lambda_s)(c_{k-2} - \lambda_{k-1})}{(c_{k-2} - \lambda_s)(\lambda_1 - c_k)} = \prod_{j=1}^{k-2} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j},$$

   where all the ratios in the product are positive numbers by (6). Consequently, the existence of the components z_j of the unit vector z in Theorem 1 depends on the necessary conditions (ii)–(iii), which can be written equivalently

   $$\frac{(c_k - \lambda_s)(c_{k-2} - \lambda_{k-1})}{(c_{k-2} - \lambda_s)(\lambda_1 - c_k)} \in \mathbb{R}^+ \quad \text{and} \quad \frac{(c_k - \lambda_s)(c_{k-2} - \lambda_{k-1})}{(c_{k-2} - \lambda_s)(\lambda_1 - c_k)} = \prod_{j=1}^{k-2} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j}.$$

2. By (6) it is clear that the ratios of the quantities $c_i - \lambda_{i+1}$ and $\lambda_i - c_i$, $i = 1,2,\ldots,k$, which are presented in [9, Theorem 3] do not need absolute value.

Example 3. Let $A = diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = diag(-1 + 10\mathbf{i}, -2 - 3\mathbf{i}, 2 - \mathbf{i}, 3 + 2\mathbf{i}, 2\mathbf{i})$, $\lambda_5 = 2\mathbf{i}$ lies on the interior of $NR[A]$. Suppose that $B = diag(-1 + 10\mathbf{i}, -2 - 3\mathbf{i}, 4, 1, 2\mathbf{i})$, where $\lambda_s = \langle\lambda_2, \lambda_3\rangle \cap \langle\lambda_4, \lambda_1\rangle \equiv 4$ and the next diagonal element $\mu = 1$ is an arbitrary interior point of $NR[A]$. Let the vector

$$\upsilon = \sqrt{\frac{12}{35}}e_1 + \sqrt{\frac{8}{35}}e_2 + \sqrt{\frac{10}{35}}e_3 + \sqrt{\frac{5}{35}}e_4.$$

It is clear to see that υ is a unit vector and $\upsilon^*A\upsilon$ is an interior point of $NR[A]$; then by (5) we have: $c_1 = \frac{-8+11\mathbf{i}}{5},\ c_2 = \frac{-2-19\mathbf{i}}{9},\ c_3 = \frac{8+3\mathbf{i}}{3},\ c_4 = \frac{31+74\mathbf{i}}{17}$. Since,

$$\frac{(c_4 - \lambda_s)(c_2 - \lambda_3)}{(\lambda_1 - c_4)(c_2 - \lambda_s)} = \frac{185}{456} \neq \prod_{j=1}^{2} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j} = \frac{(c_1 - \lambda_2)(c_2 - \lambda_3)}{(\lambda_1 - c_1)(\lambda_2 - c_2)} = \frac{5}{6},$$

according to comment 1 in Remark 2, the vector z is not defined, consequently the isometry matrix Q cannot be constructed and the corresponding 2-compression Q^*BQ does not exist.

In the following, we consider a vector z as in (8) and a $(k-2)$-compression Q^*BQ of the diagonal matrix B in (7). Necessary conditions for the existence of the vector υ are presented in the next proposition.

Theorem 4. *Let $A \in \mathcal{M}_n$ be a normal matrix, where its numerical range is a k-polygon $\mathcal{P}$. Let $z \in \mathbb{C}^n$ be a unit vector as in (8) and Q^*BQ be the corresponding $(k-2)$-compression of the diagonal matrix B in (7). If for the common points $c_i \equiv c_i(z)$ as in (11)–(13) the following assumptions hold*

(i) $\partial NR[Q^*BQ] \cap \langle\lambda_i, \lambda_{i+1}\rangle = \partial NR[P^*AP] \cap \langle\lambda_i, \lambda_{i+1}\rangle = \{c_i\}$; $i = 1, 2, \ldots, k-2, k$, and
(ii) $\frac{c_i - \lambda_{i+1}}{\lambda_i - c_i} > 0$, for $i = k-2, k$, with $\lambda_{k+1} = \lambda_1$,

then there exists a unit vector $\upsilon \in \mathbb{C}^n$, for which the isometry matrix P and the corresponding $(k-1)$-compression P^*AP are constructed; the components υ_i of the unit vector υ are defined by the equations

$$|\upsilon_{i+1}|^2 = |\upsilon_1|^2 \prod_{j=1}^{i} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j}; \quad i = 1, 2, \ldots, k-2, \quad \textit{and} \quad |\upsilon_k|^2 = |\upsilon_1|^2 \frac{c_k - \lambda_k}{\lambda_1 - c_k}, \tag{17}$$

with

$$|\upsilon_1|^2 = \left(1 + \frac{c_1 - \lambda_2}{\lambda_1 - c_1} + \prod_{j=1}^{2} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j} + \cdots + \prod_{j=1}^{k-2} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j} + \frac{c_k - \lambda_k}{\lambda_1 - c_k}\right)^{-1}. \tag{18}$$

Proof. By the unit vector $z \in \mathbb{C}^n$, the $n \times (k-2)$ matrix Q is constructed such that $\partial NR[Q^*BQ] \cap \mathcal{Q} = \{c_1, c_2, \ldots, c_{k-2}, c_k\}$. For the normal matrix $A \in \mathcal{M}_n$ we consider a unit vector $\upsilon \in \mathbb{C}^n$, and the corresponding $(k-1)$-compression P^*AP, where P is the matrix in (3). Also, recall that c_i are the common boundary points $\partial NR[P^*AP] \cap \mathcal{P}$, where c_i depend on the unit vector υ and are given by (5), i.e.,

$$c_i \equiv c_i(\upsilon) = \frac{|\upsilon_{i+1}|^2\lambda_i + |\upsilon_i|^2\lambda_{i+1}}{|\upsilon_i|^2 + |\upsilon_{i+1}|^2}; \quad i = 1, 2, \ldots, k, \quad \lambda_{k+1} = \lambda_1 \quad \text{and} \quad \upsilon_{k+1} = \upsilon_1.$$

From the expressions of the common points $c_i \equiv c_i(z) \equiv c_i(\upsilon)$, for $i = 1, 2, \ldots, k-3$, in (11), (5) and (6) we conclude that the equality in (14) is well defined and we derive

$$\frac{|\upsilon_{i+1}|^2}{|\upsilon_1|^2} = \prod_{j=1}^{i} \frac{|\upsilon_{j+1}|^2}{|\upsilon_j|^2} = \prod_{j=1}^{i} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j}; \quad i = 1, 2, \ldots, k-3, \tag{19}$$

whereby a part of the formulae in (17) arises, since $|\upsilon_{i+1}|^2$ are defined only for $i = 1, 2, \ldots, k-3$. Furthermore, from the first formula in (15), the assumption $\frac{c_{k-2} - \lambda_{k-1}}{\lambda_{k-2} - c_{k-2}} > 0$, and the formula in (19) for $i = k-3$, arises

$$|\upsilon_{k-1}|^2 = |\upsilon_{k-2}|^2 \frac{c_{k-2} - \lambda_{k-1}}{\lambda_{k-2} - c_{k-2}} = |\upsilon_1|^2 \frac{c_{k-2} - \lambda_{k-1}}{\lambda_{k-2} - c_{k-2}} \prod_{j=1}^{k-3} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j} = |\upsilon_1|^2 \prod_{j=1}^{k-2} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j}. \tag{20}$$

By the assumption $\frac{c_k - \lambda_{k+1}}{\lambda_k - c_k} = \frac{\lambda_1 - c_k}{c_k - \lambda_k} > 0$, and the first formula in (16), we obtain

$$|\upsilon_k|^2 = |\upsilon_1|^2 \frac{c_k - \lambda_k}{\lambda_1 - c_k}. \tag{21}$$

Substituting the above expressions (19)–(21) in $\sum_{i=1}^{k} |\upsilon_i|^2 = 1$, we have

$$|\upsilon_1|^2 \left(1 + \frac{c_1 - \lambda_2}{\lambda_1 - c_1} + \prod_{j=1}^{2} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j} + \cdots + \prod_{j=1}^{k-2} \frac{c_j - \lambda_{j+1}}{\lambda_j - c_j} + \frac{c_k - \lambda_k}{\lambda_1 - c_k}\right) = 1,$$

whereby the expression of $|\upsilon_1|^2$ in (18) is obvious. □

Example 5. Consider the matrices A and B in Example 3 and the unit vector

$$z = \sqrt{\frac{3}{11}}e_1 + \sqrt{\frac{6}{11}}e_2 + \sqrt{\frac{2}{11}}e_3.$$

Then, the common boundary points $c_i \in \partial NR[B] \cap \partial NR[Q^*BQ]$ are:

$$c_1 = \frac{-4 + 17\mathbf{i}}{3}, \quad c_2 = \frac{10 - 3\mathbf{i}}{4}, \quad c_4 = 2 + 4\mathbf{i}.$$

Since, $\frac{c_2-\lambda_3}{\lambda_2-c_2} = -\frac{1}{9}$, according to (ii) in Theorem 4 the vector υ cannot be defined. Considering the unit vector

$$z = \sqrt{\frac{4}{11}}e_1 + \sqrt{\frac{2}{11}}e_2 + \sqrt{\frac{5}{11}}e_3,$$

the common boundary points $c_i \in \partial NR[Q^*BQ] \cap \partial NR[B] = \{c_1, c_2, c_4\}$ are computed

$$c_1 = \frac{-5 + 4\mathbf{i}}{3}, \quad c_2 = \frac{-2 - 15\mathbf{i}}{7}, \quad c_4 = \frac{11 + 50\mathbf{i}}{9},$$

and the assumptions of Theorem 4 are verified. Consequently, by (17) and (18) the components of the unit vector υ are defined, thus a unit vector can be

$$\upsilon = \sqrt{\frac{30}{89}}e_1 + \sqrt{\frac{15}{89}}e_2 + \sqrt{\frac{20}{89}}e_3 + \sqrt{\frac{24}{89}}e_4.$$

By the above unit vectors z, υ the isometry matrices Q, P are constructed as in (3), with

$$Q = \begin{bmatrix} -0.5774 & -0.5505 \\ 0.8165 & -0.3892 \\ 0 & 0.7385 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} -0.5774 & -0.4529 & -0.3528 \\ 0.8165 & -0.3203 & -0.2495 \\ 0 & 0.8321 & -0.2881 \\ 0 & 0 & 0.8546 \\ 0 & 0 & 0 \end{bmatrix},$$

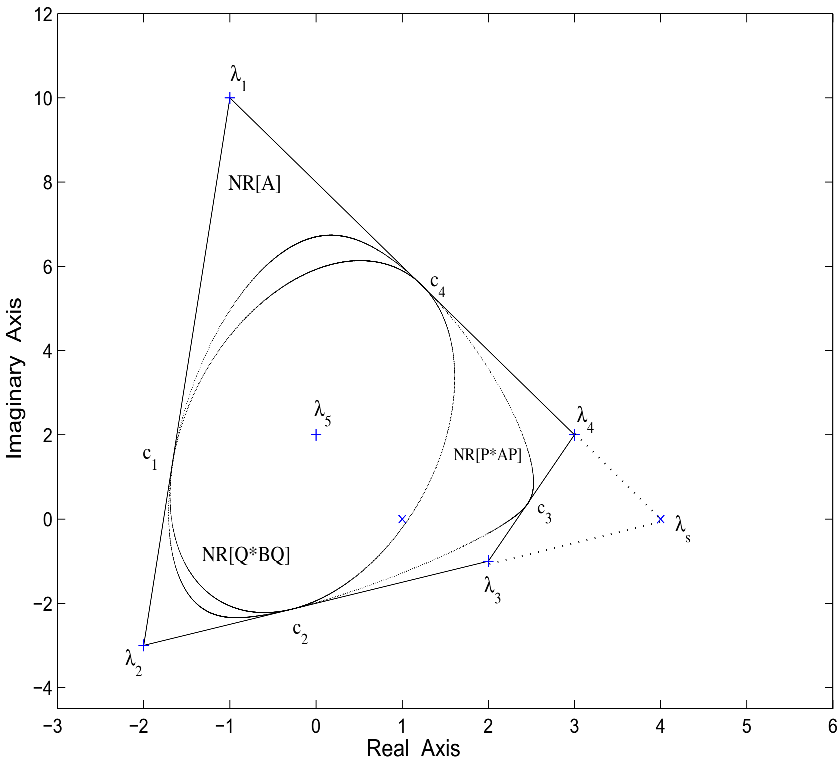

Fig. 1. The numerical ranges $NR[A]$, $NR[B]$, $NR[P^*AP]$ and $NR[Q^*BQ]$.

for which holds $\partial NR[Q^*BQ] \cap \partial NR[B] \equiv \partial NR[P^*AP] \cap \partial NR[A] = \{c_1, c_2, c_4\}$, see in Fig. 1, where the numerical ranges $NR[A]$, $NR[B]$, $NR[P^*AP]$ and $NR[Q^*BQ]$ are illustrated and the eigenvalues of A are marked with +'s and λ_4, μ with $\times$'s.

3. Properties of the boundary of the compression curve

Proposition 6. *Let $A \in \mathcal{M}_n$ be a normal matrix, whose numerical range is a k-polygon $\mathcal{P}$, and $\upsilon \in \mathbb{C}^n$ be a unit vector as in (2), P^*AP be the corresponding $(k-1)$-compression of A, where P is the isometry matrix as in (3). Let $B \in \mathcal{M}_n$ be the diagonal matrix in (7) and $z \in \mathbb{C}^n$ be the unit vector as defined in Theorem 1, when its assumptions satisfied. If the points $c_1, c_2, \ldots, c_{k-2}, c_k$ belong to an ellipse E and $s \equiv \langle \lambda_{k-1}, \lambda_k \rangle$ is a common support line of E and $\partial NR[Q^*BQ]$, where Q^*BQ is a $(k-2)$-compression of B, then*

$$\partial NR[P^*AP] \cap \partial NR[Q^*BQ] = \{c_1, c_2, \ldots, c_{k-2}, c_{k-1}, c_k\}.$$

Proof. The common support lines $s_1, s_2, \ldots, s_{k-2}, s_k$ at the points $c_1, c_2, \ldots, c_{k-2}, c_k$, and $s \equiv \langle \lambda_{k-1}, \lambda_k \rangle$ define the convex k-polygon $\mathcal{P} = \langle \lambda_1, \lambda_2, \ldots, \lambda_k \rangle$. Since the boundary of the numerical range of matrix $P^*AP \in \mathcal{M}_{k-1}$ and the ellipse E have k common support lines and $k-1$ common points, it follows that the elliptical disk of E is contained in $NR[P^*AP]$, [7]. Denoting that $c_{k-1} \in \partial NR[P^*AP] \cap s$, and s is common support line of E and $\partial NR[P^*AP]$, it follows that

$$c_{k-1} \in E. \tag{22}$$

Moreover, $\partial NR[Q^*BQ]$ and E have k common support lines and $k-1$ common points. Since the matrix $Q^*BQ \in \mathcal{M}_{k-2}$, and $k + (k-1) > 2(k-2) + 1$, the elliptical disk of E is contained in $NR[Q^*BQ]$, [7]. Therefore, by (22) and the assumption that s is a common support line of $\partial NR[Q^*BQ]$ and E, we have

$$c_{k-1} \in \partial NR[Q^*BQ],$$

hence, $c_{k-1} \in \partial NR[P^*AP] \cap \partial NR[Q^*BQ]$, that completes the proof. □

Remark 7. Consider $A \in \mathcal{M}_n$ a normal matrix, whose $NR[A]$ is quadrilateral, pentagon or hexagon, ($k = 4,5,6$), with $k \leqslant n$. The assumption of Proposition 6, that the points $c_1, c_2, \ldots, c_{k-2}, c_k \in E$, is not required. Indeed, according to known propositions of the projective geometry [5], for the existence and the geometric construction of an ellipse one from the following statements is needed:

(a) three non-collinear points and two support lines drawn through any two of these points define a unique ellipse,
(b) four different points (any three non-collinear points) and a support line drawn through any one of these points define a unique ellipse,
(c) five different points (any three non-collinear points) define a unique ellipse.

Thus, when $k = 4, 5, 6$, the common boundary points $\{c_1, c_2, \ldots, c_{k-2}, c_k\} \in \partial NR[P^*AP] \cap NR[Q^*BQ]$ and the common support lines s_1, s_2, s_{k-2}, s_k define always the ellipse E with the requirement properties of Proposition 6. Therefore, $\partial NR[P^*AP]$ and $\partial NR[Q^*BQ]$ are inscribed in $\partial NR[A]$ at the points $\{c_1, c_2, \ldots, c_{k-2}, c_{k-1}, c_k\}$, when the edge $\langle \lambda_{k-1}, \lambda_k \rangle$ is a common support line of E and $\partial NR[Q^*BQ]$.

Recall that a matrix A is called *reducible*, if it is unitarily equivalent to the direct sum of two other matrices; otherwise, A is called *irreducible*.

The following proposition consists of a generalization of a corresponding result for n distinct eigenvalues of a diagonal matrix $D \in \mathcal{M}_n$, which was presented in [8, Theorem 3.1].

Theorem 8. *Let $A \in \mathcal{M}_n$ be a normal matrix, whose numerical range is a k-polygon with $k \leqslant n$. Let P be the isometry matrix as in (3) and $C = P^*AP$ be a $(k-1)$-compression of A. Then,*

(i) $\partial NR[C]$ contains no line segment,
(ii) C is irreducible,
(iii) $\partial NR[C]$ is a differentiable curve.

Proof.

(i) By [1, Theorem 1], for the normal matrix A with $NR[A] = \langle \lambda_1, \lambda_2, \ldots, \lambda_k \rangle = \mathcal{P}$ and for a unit vector υ the isometry matrix P is constructed as in (3), and the common points c_i of $\partial NR[C] \cap \mathcal{P}$ are given by (5). Suppose that $\langle a, b \rangle$ is a line segment on $\partial NR[C]$, which belongs on part of boundary between c_i and c_{i+1}, for $i = 1, 2, \ldots, k$, and the vectors $w_i, w_{i+1} \in \mathbb{C}^{k-1}$ are defined by

$$Pw_i = y_i \in E^{\perp}_{W_x}(\upsilon), \quad \text{and} \quad Pw_{i+1} = y_{i+1} \in E^{\perp}_{W_x}(\upsilon),$$

with $y_i, y_{i+1} \in \mathbb{C}^n$ the unit vectors in (4). Note that $\|w_i\| = \|w_{i+1}\| = 1$, due to $P^*P = I_{k-1}$, and $w_r^* w_r = w_r^* P^* P w_r = y_r^* y_r = 1$, for $r = i, i + 1$. Consider the subspace $K \subseteq \mathbb{C}^n$, which is spanned by the vectors Pw, with $w \in \mathbb{C}^{k-1}$, $\|w\| = 1$, and such that $w^*P^*APw = c$, for some $c \in \langle a,b\rangle$, and the subspace $L \subseteq \mathbb{C}^n$, which is spanned by the linearly independent vectors $Pw_1, Pw_2, \ldots, Pw_{i-1}, Pw_{i+2}, \ldots, Pw_k$. Obviously, $dim\ L = k - 2$, and $dim\ K \geqslant 2$, since K contains at least the two vectors, which correspond at the points c_i, c_{i+1}. Thus,

$$dim(K \cap L) = dimK + dimL - dim(K + L) \geqslant 2 + (k-2) - (k-1) = 1,$$

consequently, there exists at least a vector $Pw_0 \in K \cap L$, with $\|w_0\| = 1$, and $w_0 \in \mathbb{C}^{k-1}$.
Consider $Pw_0 \in L$, due to definition L,

$$Pw_0 = \sum_{j=1}^{k} a_j P w_j, \quad \text{for suitable } a_j, \quad \text{and} \quad j \neq i, i+1,$$

whereby arises

$$w_0^* P^* A P w_0 = \left(\sum_{j \neq i, i+1}^{k} a_j P w_j \right)^* \sum_{j \neq i, i+1}^{k} a_j A P w_j,$$

i.e., the point $w_0^* P^* A P w_0 \in \langle \lambda_1, \lambda_2, \ldots, \lambda_i, \lambda_{i+2}, \ldots, \lambda_k \rangle$.
Moreover, $Pw_0 \in K \Rightarrow w_0^* P^* A P w_0 = c \in \langle a, b \rangle$. This is a contradiction, because of $\langle a,b\rangle$ lies on the part of $\partial NR[C]$ between c_i, c_{i+1}, and outer of the polygon $\langle \lambda_1, \lambda_2, \ldots, \lambda_i, \lambda_{i+2}, \ldots, \lambda_k \rangle$. Hence, $\partial NR[C]$ does not include line segment.

(ii) Suppose that the matrix $C \in \mathcal{M}_{k-1}$ is reducible, thus, there exists a unitary matrix $U \in \mathcal{M}_{k-1}$ such that

$$U^* C U = U^* P^* A P U = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}.$$

Since $NR[C] = NR[U^*CU] = \mathrm{Co}\{NR[A_1] \cup NR[A_2]\}$, [11], for any $c_i \in \partial NR[C]$, there exists $t \in [0,1]$, such that $c_i = ta_1 + (1 - t)a_2$, with $a_1 \in NR[A_1]$, and $a_2 \in NR[A_2]$, for $i = 1,2,\ldots,k$. Moreover, since the boundary of $NR[C]$ does not include line segment by (i) and $c_i \in \partial NR[C]$ one of the following statements holds :

(a) $t = 0 \Rightarrow c_i = a_2 \in NR[A_2]$, or
(b) $t = 1 \Rightarrow c_i = a_1 \in NR[A_1]$, or
(c) $a_1 = a_2 \Rightarrow c_i = a_2 \in NR[A_2]$.

Therefore, for $i = 1,2,\ldots,k$, either $c_i \in NR[A_1]$ or $c_i \in NR[A_2]$, and $c_i \notin NR[A_1] \cap NR[A_2]$. In the following, consider two different points c_1, c_2, that lie on the numerical ranges of A_1, A_2.Without loss of generality, suppose $c_1 \in NR[A_1]$ and $c_2 \in NR[A_2]$, which we will prove that is a contradiction. Since $c_1 \in NR[A_1]$, there exists a unit vector u, such that $c_1 = u^*A_1u$, and

$$c_1 = [\,u^* \quad 0\,] U^* P^* A P U \begin{bmatrix} u \\ 0 \end{bmatrix} = w_1^* P^* A P w_1,$$

where $w_1 = U\begin{bmatrix} u \\ 0 \end{bmatrix} \in \mathbb{C}^{k-1}$. Similarly, if $c_2 \in NR[A_2]$, there exists a unit vector v, such that $c_2 = v^*A_2v$, and

$$c_2 = [\,0 \quad v^*\,] U^* P^* A P U \begin{bmatrix} 0 \\ v \end{bmatrix} = w_2^* P^* A P w_2,$$

where $w_2 = U\begin{bmatrix} 0 \\ v \end{bmatrix} \in \mathbb{C}^{k-1}$. Let $Pw_1 \equiv y_1$, $Pw_2 \equiv y_2$, for y_1, y_2 as in (4), i.e.,

$$Pw_1 \equiv y_1 = \frac{\bar{\upsilon}_2}{\sqrt{|\upsilon_1|^2 + |\upsilon_2|^2}} x_1 - \frac{\bar{\upsilon}_1}{\sqrt{|\upsilon_1|^2 + |\upsilon_2|^2}} x_2,$$

$$Pw_2 \equiv y_2 = \frac{\bar{\upsilon}_3}{\sqrt{|\upsilon_2|^2 + |\upsilon_3|^2}} x_2 - \frac{\bar{\upsilon}_2}{\sqrt{|\upsilon_2|^2 + |\upsilon_3|^2}} x_3.$$

By the two last expressions we conclude

$$(Pw_2)^* P w_1 = -\frac{\bar{\upsilon}_1 \upsilon_3}{\sqrt{|\upsilon_1|^2 + |\upsilon_2|^2}\sqrt{|\upsilon_2|^2 + |\upsilon_3|^2}} \neq 0, \tag{23}$$

since $\upsilon_i \neq 0$, $i = 1,2,\ldots,k$, (see (2)). Also, using the above vectors w_1, w_2 it is clear to see that Pw_1, Pw_2 are orthonormal, which is a contradiction from (23). Therefore, $c_2 \in NR[A_1]$, i.e., both c_1, $c_2 \in NR[A_1]$.
Similarly, it is proved that, $c_i \in NR[A_1]$, for every $i = 1,2,\ldots,k$. Moreover, the unit vectors $Pw_1 \equiv y_1$, $Pw_2 \equiv y_2, \ldots, Pw_{k-1} \equiv y_{k-1}$ consist of a basis of $E^{\perp}_{W_x}(\upsilon)$, whereby it is implied $A_1 \in \mathcal{M}_{k-1}$, consequently, C is an irreducible matrix.

(iii) The curve of the boundary of $NR[C]$ is differentiable, otherwise, if there exists a point, where it is not differentiable, then at this point the curve has to present:
   (a) either sharp point, this case is contradiction by (i), or
   (b) the matrix C has normal eigenvalues [11, Theorem 1.6.6], but in this case the matrix C has to be reducible, that is impossible from (ii). □

In the following proposition a condition is formulated such that the boundary of the numerical range of a 3×3 matrix is an ellipse, when the matrix is a suitable 3-compression of a normal matrix A, whose numerical range is a quadrilateral. The equation

$$det(uH_A + vS_A + \omega I) = 0,$$

with u, v, ω viewed as homogeneous line coordinates, defines an algebraic curve of class n. The real part of this curve is called *associated curve* of A and denoted by $\mathcal{C}_A$. The eigenvalues of A are the real foci of $\mathcal{C}_A$, and $NR[A] = \mathrm{Co}\{\mathcal{C}_A\}$, (see [12]).

Proposition 9. *Let $A \in \mathcal{M}_n$ be a normal matrix, where $NR[A] = \langle\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle$, and let $z = z_1e_1 + z_2e_2 + z_3e_3 \in \mathbb{C}^n$ be a unit vector as in (8), which corresponds to 2-compression Q^*BQ of the diagonal matrix $B = diag(\lambda_1, \lambda_2, \lambda_s, \mu, \lambda_5, \ldots, \lambda_n)$ as in (7), and $c_1, c_2, c_4 \in \partial NR[Q^*BQ]$ as in Theorem 4. If the line*

$$(Im\lambda_3 - Im\lambda_4)x + (Re\lambda_4 - Re\lambda_3)y + \omega = 0 \tag{24}$$

is a support line of the elliptical disk $NR[Q^*BQ]$, with $\omega = Re\lambda_3 Im\lambda_4 - Re\lambda_4 Im\lambda_3$, then

$$NR[Q^*BQ] = NR[P^*AP],$$

where P is the $n\times 3$ matrix as in (3) corresponding to the unit vector $v \in \mathbb{C}^n$ defined in (17) and (18).

Proof. Since $Q^*BQ$is a 2-compression of B, $\partial NR[Q^*BQ]$ is an ellipse, [11]. According to Fiedler' s comments in [6] and the assumption in (24), we conclude

$$det[(Im\lambda_3 - Im\lambda_4)H_{Q^*BQ} + (Re\lambda_4 - Re\lambda_3)S_{Q^*BQ} + \omega I_2] = 0,$$

and the support line intersects $\partial NR[Q^*BQ]$ at the point q^*Q^*BQq, where q is a unit eigenvector of the Hermitian matrix $(Im\lambda_4 - Im\lambda_3)H_{Q^*BQ} + (Re\lambda_3 - Re\lambda_4)S_{Q^*BQ}$, corresponding to the eigenvalue $\omega = Re\lambda_3 Im\lambda_4 - Re\lambda_4 Im\lambda_3$. Since $\partial NR[Q^*BQ] \cap \partial NR[P^*AP] = \{c_1, c_2, c_4\}$, and the line segment $\langle\lambda_3, \lambda_4\rangle$ lies on line in (24), the Eq. (24) is common support line of $\partial NR[Q^*BQ]$ and $\partial NR[P^*AP]$. Moreover, according to (a) in Remark 7 a unique ellipse is defined, $\partial NR[Q^*BQ]$. Hence, by Proposition 6, we conclude that $c_3 = q^*Q^*B\ Q\ q \in \partial NR[P^*AP] \cap \partial NR[Q^*BQ]$. Also, the boundaries of the numerical range of the matrix $P^*AP \in \mathcal{M}_3$ and of the elliptical disk $NR[Q^*BQ]$ have four common points and four common support lines, thus

$$NR[Q^*BQ] \subseteq NR[P^*AP] \quad \text{with} \quad \sigma(P^*AP) = \{\epsilon_1, \epsilon_2\}, \tag{25}$$

where ϵ_1, ϵ_2 are the foci of ellipse, $(\epsilon_1 \neq \epsilon_2)$, [7]. According to a result of Kippenhahn (see [12, sect. 7, Theorem 26]), the associated curve $\mathcal{C}_{P^*AP}$

(a) consists of three points, when $NR[P^*AP]$ is the convex hull of these points, but this case is impossible by Theorem 8 (i), either
(b) the degree of $\mathcal{C}_{P^*AP}$ is equal to 4, when $\partial NR[P^*AP]$ has a flat portion, which is impossible by Theorem 8 (i), or
(c) the degree of curve is equal to 6, when $\sigma\ (P^*AP)$ has three distinct eigenvalues, which is a contradiction by (25), or
(d) consists of a point and an ellipse. We claim that the point is in the interior of elliptical disk; otherwise $\partial\ NR[P^*AP]$ presents sharp point or the point is a normal eigenvalue of P^*AP, both of which are impossible by Theorem 8 (i), (ii). Therefore, the curve $\mathcal{C}_{P^*AP}$ consists of an ellipse, i.e., $NR[P^*AP]$ is an elliptical disk.

Since $\partial NR[Q^*BQ]$ and $\partial NR[P^*AP]$ have the same common points and support lines, the two elliptical disks are identical. □

Example 10. Let $A = diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$, $B = diag(\lambda_1, \lambda_2, \lambda_s, \mu, \lambda_5)$, with

$$A = diag\left(-2\sqrt{2} + \left(2 + 2\sqrt{2}\right)\mathbf{i}, -2\sqrt{2} - \left(2 + 2\sqrt{2}\right)\mathbf{i},\ 2\sqrt{2} - \left(2\sqrt{2} - 2\right)\mathbf{i},\ 2\sqrt{2} + \left(2\sqrt{2} - 2\right)\mathbf{i},\ \mathbf{i}\right),$$
$$B = diag\left(-2\sqrt{2} + (2 + 2\sqrt{2})\mathbf{i},\ -2\sqrt{2} - \left(2 + 2\sqrt{2}\right)\mathbf{i},\ 4,\ 2\sqrt{2},\ \mathbf{i}\right),$$

and the unit vector $z = \sqrt{\frac{1}{1+\sqrt{2}}}e_1 + \sqrt{\frac{1}{1+\sqrt{2}}}e_2 + \frac{1}{1+\sqrt{2}}e_3$, whereby an isometry matrix Q and the corresponding 2-compression are computed

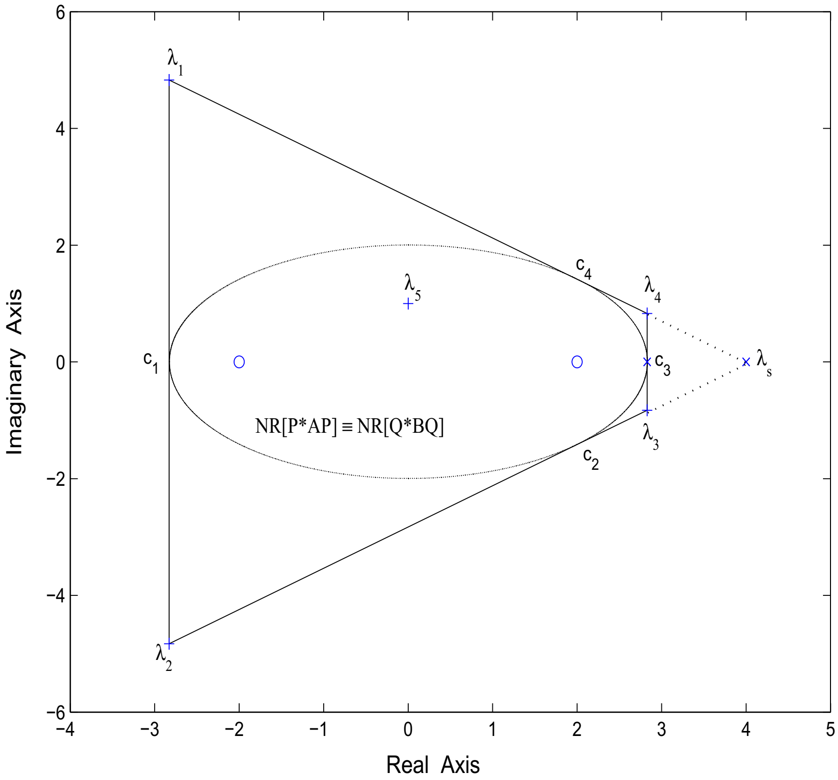

Fig. 2. The numerical ranges $NR[P^*AP]$, $NR[Q^*BQ]$ are identical.

$$Q = \begin{bmatrix} -1/\sqrt{2} & -0.2929 \\ 1/\sqrt{2} & -0.2929 \\ 0 & 0.9102 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q^*BQ = \begin{bmatrix} -2\sqrt{2} & 2\mathbf{i} \\ 2\mathbf{i} & 2\sqrt{2} \end{bmatrix},$$

with $\sigma(Q^*BQ) = \{-2, 2\}$, $tr(Q^*BQ) = 0$; the length of the major axis is equal to $4\sqrt{2}$ and of the minor axis 4 (see Notations and preliminaries in Section 1), thus $\partial NR[Q^*BQ] = \{(x,y) : \frac{x^2}{8} + \frac{y^2}{4} = 1\}$. Clearly, the line $x = 2\sqrt{2}$ is a support line of $\partial\, NR[Q^*BQ]$ at the point $c_3 = 2\sqrt{2}$. The common points of $\partial NR[Q^*BQ] \cap \partial NR[B] \cap \partial NR[A]$ are : $c_1 = -2\sqrt{2},\ c_2 = 2 - \sqrt{2}\mathbf{i},\ c_4 = 2 + \sqrt{2}\mathbf{i}$ Since $\frac{c_2-\lambda_3}{\lambda_2-c_2} = \frac{1}{(1+\sqrt{2})^2}, \ \frac{c_4-\lambda_1}{\lambda_4-c_4} = \frac{1}{(\sqrt{2}-1)^2}$, the unit vector $\upsilon$is defined by (17) and (18)

$$\upsilon = \sqrt{\frac{2+\sqrt{2}}{8}}e_1 + \sqrt{\frac{2+\sqrt{2}}{8}}e_2 + \sqrt{\frac{2-\sqrt{2}}{8}}e_3 + \sqrt{\frac{2-\sqrt{2}}{8}}e_4.$$

The spectrum of 3-compression P^*AP is $\sigma(P^*AP) = \{-2, 2\}$, and the point $c_3 = 2\sqrt{2}$ is computed by (5) for the above vector υ, thus $c_3 \in \partial NR[A] \cap \partial NR[P^*AP] = \partial NR[A] \cap \partial NR[Q^*BQ]$. In Fig. 2 the numerical ranges $NR[A]$, $NR[B]$, $NR[P^*AP]$ and $NR[Q^*BQ]$ are illustrated, the points λ_s, μ are marked with $\times$'s, the eigenvalues of A are marked with +'s and the eigenvalues of P^*AP with o's.

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