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On the Hermitian solutions of the matrix equation $X^s + A^*X^{-s}A = Q$

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Abstract

In this paper, necessary and sufficient conditions for the existence of the Hermitian solutions of the nonlinear matrix equation $X^s+A^*X^{-s}A = Q$ are presented, when A is a nonsingular matrix and s an integer. The formulas for the computation of these solutions are presented. An algebraic method for the computation of the solutions is proposed; the method is based on the algebraic solution of the corresponding discrete time Riccati equation. The exact number of the Hermitian solutions is also derived. The formula for the computation of the maximal solution of the matrix equation $X^s - A^*X^{-s}A = Q$ is given as an application of the formulas derived for solving $X^s + A^*X^{-s}A = Q$. The results are verified through simulation experiments.

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1 Introduction

The central issue of this paper is the investigation and the computation of the Hermitian solutions of the matrix equation

$$X^s + A^*X^{-s}A = Q, \tag{1}$$

where the nonsingular matrix $A \in \mathcal{M}_n$, $\mathcal{M}_n$ denotes the set of all $n \times n$ matrices with complex or real entries, A^* stands for the conjugate transpose of A, $Q \in \mathcal{M}_n$ is a Hermitian positive definite matrix and s is an integer. The matrix equation of the form (1) arises in many applications in various research areas including control theory, ladder networks, dynamic programming, stochastic filtering and statistics; see [2, 8, 12] and the references given therein. In the case that A is nonsingular and $s = 1$, necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation (1) have been investigated by many authors [2, 3, 7, 8, 12] and formulas for the computation of Hermitian and non-Hermitian solutions can be found in [3].

The study *only* of the positive definite solutions of the nonlinear matrix equation (1) has been presented in [1] and the more general equation $X^s + A^*X^{-t}A = Q$, with A nonsingular, has been considered in [4, 6, 11] (see also references therein) for s, t positive integers and in [5] when $s \geq 1, 0 < t \leq 1$ and $0 < s \leq 1, t \geq 1$; there some existence conditions and properties of its positive definite solutions are obtained.

To describe our results, we introduce some notations and definitions. For a Hermitian matrix $A \in \mathcal{M}_n$, the notation $A > 0\,(A \geq 0)$ means that A is a positive definite (semidefinite) matrix and for the Hermitian matrices $A, B \in \mathcal{M}_n$, the notation $A > B\,(A \geq B)$ means that $A - B > 0\,(A - B \geq 0)$. For $A \in \mathcal{M}_n$, $\lambda_i(A)$ denotes an eigenvalue of A, $\sigma(A)$ the spectrum of A,

$$\begin{aligned} \rho(A) &= \max\{|\lambda_i(A)| \ : \ \lambda_i(A) \in \sigma(A)\} \\ \|A\| &= \max\{\sqrt{\lambda_i(A^*A)} \ : \ \lambda_i(A^*A) \in \sigma(A^*A)\} \end{aligned} \tag{2}$$

denotes the *spectral radius* and *spectral norm* of A, respectively, and

$$r(A) = \max\{|x^*Ax| \ : \ \text{for each vector } x \in \mathbb{C}^n, \ \text{with } x^*x = 1\} \tag{3}$$

denotes the *numerical radius* of A. It is also known [2, 3, 8, 12] that, when $X > 0$ is a solution of $X + A^*X^{-1}A = Q$, then there exist minimal and maximal solutions X_{min} and X_{max}, respectively, such that $0 < X_{min} \leq X \leq X_{max}$ for any solution $X > 0$. The minimal and maximal solutions are referred as the *extreme solutions*. Moreover, the existence of a positive definite solution depends on the *numerical radius* of the matrix $Q^{-1/2}AQ^{-1/2}$, [8, Theorem 5.2].

Note that, setting in (1)

$$Y = X^s \tag{4}$$

we obtain

$$Y + A^*Y^{-1}A = Q. \tag{5}$$

In this paper, we focus on the computation of the Hermitian solutions of the matrix equation (1), with $A \in \mathcal{M}_n$ a nonsingular matrix, $Q > 0$ and s an integer with $s \geq 1$. In Section 2, specific necessary and sufficient conditions for the existence of positive definite solutions of (1) are presented and formulas for computing the Hermitian solutions of (1) are given. In Section 3, an algebraic method for computing the Hermitian solutions is proposed; the method is based on the algebraic solution of the corresponding discrete time Riccati equation. The number of Hermitian solutions of (1) is also derived (if such solutions exist). In Section 4, the Hermitian solutions of the matrix equation $X^s - A^*X^{-s}A = Q$ are discussed and the maximal solution is formulated as an implementation of the results for solving $X^s + A^*X^{-s}A = Q$. In Section 5, simulation results are given to illustrate the efficiency of the proposed results.

2 Existence and formulas of Hermitian solutions of $X^s + A^*X^{-s}A = Q$

In the first part of this section, in order to present necessary and sufficient conditions for the existence of Hermitian solutions of the nonlinear matrix equation (1), we utilize the properties of numerical radius and the following proposition.

Proposition 2.1. [1, Theorem 2] *Let $A \in \mathcal{M}_n$ be a nonsingular matrix and $Q \in \mathcal{M}_n$ with $Q > 0$. The equation (1) has a positive definite solution $X \in \mathcal{M}_n$ if and only if for the numerical radius of $Q^{-1/2}AQ^{-1/2}$ holds*

$$r(Q^{-1/2}AQ^{-1/2}) \leq \frac{1}{2}. \tag{6}$$

It is well known that for a Hermitian matrix A holds $\|A\| = \rho(A)$ and by [10, p. 12] and the definitions of radius in (2), (3) we conclude :

$$r(A) = \max\{|\lambda_{min}(A)|, |\lambda_{max}(A)|\} = \rho(A)$$

Thus, we have the equality:

$$r(A) = \rho(A) = \|A\| \tag{7}$$

Lemma 2.1. [10, p. 44] *Suppose that $A \in \mathcal{M}_n$ is a matrix with nonnegative real entries. Then $r(A) = \rho(H_A)$, where $H_A = \frac{A+A^*}{2}$ denotes the Hermitian part of A.*

Proposition 2.2. *Suppose that $Q^{-1/2}AQ^{-1/2} \in \mathcal{M}_n$ has nonnegative real entries and $A \in \mathcal{M}_n$ is a nonsingular matrix. The equation (1) has at least one positive definite solution if and only if*

$$\|Q^{-1/2}H_AQ^{-1/2}\| \leq \frac{1}{2}, \tag{8}$$

where $H_A = \frac{1}{2}(A + A^*)$.

Proof. Since H_A is Hermitian and $Q^{-1/2} > 0$, it follows that $Q^{-1/2}H_AQ^{-1/2}$ is Hermitian as well, thus from (7) arises:

$$\|Q^{-1/2}H_AQ^{-1/2}\| = \rho(Q^{-1/2}H_AQ^{-1/2}) \tag{9}$$

Moreover, from Lemma 2.1 we derive

$$\begin{aligned} r(Q^{-1/2}AQ^{-1/2}) &= \rho(H_{Q^{-1/2}AQ^{-1/2}}) \\ &= \rho\left(\frac{1}{2}(Q^{-1/2}(A + A^*)Q^{-1/2})\right) = \rho(Q^{-1/2}H_AQ^{-1/2}). \end{aligned}$$

Combining the last relation, (6) and (9) in Proposition 2.1 we derive (8). □

In the second part of this section, the formulas of all Hermitian solutions of (1) are given by the following theorem, the existence of which is guaranteed by Proposition 2.1.

Theorem 2.3. *Let $A \in \mathcal{M}_n$ be a nonsingular matrix with $r(Q^{-1/2}AQ^{-1/2}) \leq \frac{1}{2}$ and $Y \in \mathcal{M}_n$ be a positive definite solution of the equation (5). For each Y, there exist a unitary matrix $U \in \mathcal{M}_n$, which diagonalizes Y, and a Hermitian solution $X \in \mathcal{M}_n$ of the matrix equation (1), such that*

(i) *if $s = 2\ell + 1$, $\ell = 0, 1, \ldots,$ then*

$$X = U diag\left(\sqrt[s]{\lambda_1(Y)}, \sqrt[s]{\lambda_2(Y)}, \ldots, \sqrt[s]{\lambda_n(Y)}\right) U^*, \tag{10}$$

(ii) *if $s = 2\ell$, $\ell = 1, 2, \ldots,$ then*

$$X = U diag\left(\pm\sqrt[s]{\lambda_1(Y)}, \pm\sqrt[s]{\lambda_2(Y)}, \ldots, \pm\sqrt[s]{\lambda_n(Y)}\right) U^*, \tag{11}$$

where $\lambda_i(Y) \in \sigma(Y)$, $i = 1, 2, \ldots, n$, and $\pm$ in (11) is denoted all the possible combinations of the associated algebraic signs.

Proof. Since $r(Q^{-1/2}AQ^{-1/2}) \leq \frac{1}{2}$, the matrix equation (1) has at least one positive definite solution by Proposition 2.1; also, the same condition guarantees the existence of at least one positive definite solution of (5) by [8, Theorem 5.2], this solution is denoted by $Y \in \mathcal{M}_n$. According to the spectral theorem for Y, there exists a unitary matrix $U \in \mathcal{M}_n$ such that

$$Y = UDU^*, \tag{12}$$

where $D = diag(\lambda_1(Y), \ldots, \lambda_n(Y))$ and $\lambda_i(Y) \in \sigma(Y)$ are positive real numbers. Using the unitarity of $U \in \mathcal{M}_n$ and the formulas of X, Y from (10), (12), we derive :

$$\begin{aligned}
&X^s + A^*X^{-s}A \\
&\quad = \left(U diag(\sqrt[s]{\lambda_1(Y)}, \ldots, \sqrt[s]{\lambda_n(Y)}\,)U^*\right)^s \\
&\qquad + A^* \left(U diag(\sqrt[s]{\lambda_1(Y)}, \ldots, \sqrt[s]{\lambda_n(Y)}\,)U^*\right)^{-s} A \\
&\quad = U \left(diag(\sqrt[s]{\lambda_1(Y)}, \ldots, \sqrt[s]{\lambda_n(Y)}\,)\right)^s U^* \\
&\qquad + A^* \left(U^*\right)^{-1} \left(diag(\sqrt[s]{\lambda_1(Y)}, \ldots, \sqrt[s]{\lambda_n(Y)}\,)\right)^{-s} U^{-1}A \\
&\quad = UDU^* + A^* \left(U^*\right)^{-1} D^{-1}U^{-1}A \\
&\quad = Y + A^*Y^{-1}A = Q
\end{aligned}$$

Hence, X in (10) consists a solution of (1) and due to $\lambda_i(Y) > 0$ it is a positive definite solution. Similarly, it can be shown that X in (11) is a Hermitian solution of (1). □

Remark 2.1. Concluding the section we note that :

(i) Proposition 2.1 guarantees that the existence of Hermitian solutions of (1) depends on the numerical range of $Q^{-1/2}AQ^{-1/2}$ and Theorem 2.3 determines the definiteness of these solutions, which depends on s; when s is an odd number, *only* positive definite solutions are derived and formulated by (10); when s is an even number, then among the (Hermitian) solutions there exist negative definite and indefinite solutions, which are formed by (11). The simple-closed forms, which are proposed in Theorem 2.3 for computing all the Hermitian solutions of (1), generalize the results in [1, Theorem 2], that are referred only to positive definite solutions.

(ii) The Hermitian solutions in (11) are linear dependent, since these are in pairs opposite.

(iii) The condition for the existence of solutions of (1), which is related to the numerical range of $Q^{-1/2}AQ^{-1/2}$ in Proposition 2.1, is more general than the formed conditions in [6, Theorem 3.1]

$$\lambda_{max}(A^*A) < \frac{1}{4}(\lambda_{min}(Q))^2 \tag{13}$$

and in [4, Theorem 2.3]

$$\lambda_{min}(A^*A) < \frac{1}{4}(\lambda_{min}(Q))^2 \tag{14}$$

and

$$\lambda_{max}(A^*A) \leq \frac{\alpha_1^{s-1}}{2}\sqrt[s]{\frac{1}{2}\,(\lambda_{min}(Q))^{s+1}}, \tag{15}$$

where α_1 is the solution of the equation $x^{2s} - \lambda_{max}(Q)x^s + \lambda_{min}(A^*A) = 0$, that lies in the interval $\left(0, \sqrt[s]{\frac{1}{2}\,\lambda_{min}(Q)}\right)$; when (13) holds or the conditions (14) and (15) are satisfied, then some positive definite solutions are described in [4, 6] through matrix sets but not through formulas. Notice that in Example 2 and for $s = 3$, the associated matrix equation has Hermitian solutions, although only the inequality in (14) is verified.

3 Computation of Hermitian solutions of $X^s + A^* X^{-s} A = Q$

Let $r(Q^{-1/2}AQ^{-1/2}) \leq \frac{1}{2}$ and Y be a positive definite solution of (5). Working as in [2]-[3], we are able to derive a Riccati equation, which is equivalent to the matrix equation (5). In particular, (5) can be written as

$$Y = Q - A^* Y^{-1} A,$$

whereby the following equivalent equation arises :

$$\begin{aligned} Y &= Q - A^* \left[Q - A^* Y^{-1} A\right]^{-1} A = Q - A^* A^{-1} \left[(A^*)^{-1} Q A^{-1} - Y^{-1}\right]^{-1} (A^*)^{-1} A \\ &= Q + A^* A^{-1} \left[Y^{-1} + (-(A^*)^{-1} Q A^{-1})\right]^{-1} \left(A^* A^{-1}\right)^* \end{aligned}$$

Substituting in the above equation

$$F = A^* A^{-1} \quad \text{and} \quad G = -(A^*)^{-1} Q A^{-1} \tag{16}$$

the related discrete time Riccati equation is derived :

$$Y = Q + F\left(Y^{-1} + G\right)^{-1} F^* \tag{17}$$

Therefore (5) is equivalent to the related discrete time Riccati equation (17), for the solution of which one may use the *algebraic Riccati Equation Solution Method*, [2, 3].

Specifically, from the Riccati equation's parameters in (16), the following matrix is formed

$$\Phi^+ = \begin{bmatrix} A^{-1}A^* & -A^{-1}QA^{-1} \\ QA^{-1}A^* & A^*A^{-1} - QA^{-1}QA^{-1} \end{bmatrix}, \tag{18}$$

which satisfies $(\Phi^+)^* J \Phi^+ = J$, where $J = \begin{bmatrix} \mathbb{O} & -I \\ I & \mathbb{O} \end{bmatrix}$, i.e., Φ^+ is a symplectic matrix. All the eigenvalues of Φ^+ are non-zero ($0 \notin \sigma(\Phi^+)$) and it may be diagonalized in the form

$$\Phi^+ = WLW^{-1},$$

where L is a $2n \times 2n$ diagonal matrix with diagonal entries the eigenvalues of Φ^+

$$L = \begin{bmatrix} L_1 & \mathbb{O} \\ \mathbb{O} & L_2 \end{bmatrix} \tag{19}$$

and W contains the corresponding eigenvectors

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}. \tag{20}$$

In the case that (5) is solvable, then all its solutions Y_j, $j = 1, 2, \ldots, 2n$ are given by the formulas:

$$Y_j = W_{21}\left(W_{11}\right)^{-1} \quad \text{and} \quad Y_j = W_{22}\left(W_{12}\right)^{-1}, \tag{21}$$

where the block matrices $W_{11}, W_{12}, W_{21}, W_{22}$ are defined by the partition of W in (20) following every different permutation of its column [3, Propositions 1,2]. Among Y_j there exist positive definite solutions for some $j = 1, 2, \ldots, 2n$, therefore Theorem 2.3 can be applied, i.e., there exist unitary matrices U_j, and $D = diag(\lambda_1(Y_j), \ldots, \lambda_n(Y_j)) > 0$, with $\lambda_i(Y_j) \in \sigma(Y_j)$ such that the solutions of (1) depend on s; in particular the positive definite solutions are given by (10) and the Hermitian solutions are formulated by (11).

We remind that Φ^+ is a symplectic matrix and its eigenvalues occur in reciprocal pairs. Therefore, we may arrange them in the diagonal matrix L in (19) so that L_1 contains all the eigenvalues of Φ^+ lying outside the unit circle, and $L_2 = \left(L_1\right)^{-1}$. The above process defines a corresponding (special) arrangement of eigenvectors of Φ^+ in W, which we denote by

$$\hat{W} = \begin{bmatrix} \hat{W}_{11} & \hat{W}_{12} \\ \hat{W}_{21} & \hat{W}_{22} \end{bmatrix}.$$

Using the matrix $\hat{W}$, the unique positive definite solutions of the discrete time Riccati equation in (17) coincide with *the extreme solutions* of (5) and these are formed :

$$Y_{max} = \hat{W}_{21}(\hat{W}_{11})^{-1} > 0 \quad \text{and} \quad Y_{min} = \hat{W}_{22}(\hat{W}_{12})^{-1} > 0 \tag{22}$$

Moreover, it is well known [2, 3] that, when $A \in \mathcal{M}_n$ is nonsingular, the existence of a finite number of positive definite solutions of the matrix equation (5) depends on the eigenvalues of the matrix Φ^+. In the following, $V(\lambda_i(\Phi^+))$ denotes the eigenspace corresponding to eigenvalue $\lambda_i(\Phi^+)$ and $dim(V(\lambda_i(\Phi^+)))$ denotes its dimension.

Theorem 3.1. *Let $A \in \mathcal{M}_n$ be a nonsingular matrix, Φ^+ be the matrix in (18) and assume that the matrix equation $X^s + A^* X^{-s} A = Q$ has at least one positive definite solution. If $dim(V(\lambda_i(\Phi^+))) = 1$ for every eigenvalue $\lambda_i(\Phi^+)$, $i = 1, 2, \ldots, 2n$, with $|\lambda_i(\Phi^+)| \neq 1$, then there exists a finite number of Hermitian solutions (h.s.) of (1). In particular, when s is an even number, then the number of solutions is equal to*

$$\# \, h.s. = 2^n \prod_{j=1}^{m} (n_j + 1) \tag{23}$$

and when s is an odd number, then the number of positive definite solutions (h.p.d.s.) is given by

$$\# \, h.p.d.s. = \prod_{j=1}^{m} (n_j + 1). \tag{24}$$

If A, Q are real matrices, then among the h.s. and h.p.d.s. there exist real symmetric solutions (r.s.s.) (or real positive symmetric solutions (r.p.s.s.)) of (1) with

$$\# \, r.s.s. = 2^n \prod_{k=1}^{p+q} (n_k + 1); \quad for \;\; s = 2\ell, \;\; \ell = 1, 2, \ldots \tag{25}$$

$$\# \, r.p.s.s. = \prod_{k=1}^{p+q} (n_k + 1); \quad for \;\; s = 2\ell + 1, \;\; \ell = 0, 1, \ldots \tag{26}$$

where m is the number of the distinct eigenvalues of Φ^+, that lie outside the unit circle, with algebraic multiplicity n_j, $j = 1, 2, \ldots, m$, p is the number of real distinct eigenvalues of Φ^+ lying outside the unit circle, q is the number of complex conjugate pairs of eigenvalues lying outside the unit circle, with algebraic multiplicity n_k, $k = 1, 2, \ldots, p + q$.

Proof. Assume that $X \in \mathcal{M}_n$ is a positive definite solution of (1). According to Theorem 2.3, every positive definite solution $X \in \mathcal{M}_n$ of (1) is related to a positive definite solution $Y \in \mathcal{M}_n$ of (5) by (4). Hence, the unique symplectic matrix Φ^+ in (18) corresponds to the two matrix equations (5) and (1). Thus, when $s = 2\ell + 1$, for $\ell = 0, 1, \ldots$, the total number of positive definite (real positive symmetric) solutions of (1) is the same as in [1, 3] and is given by (24), (26), respectively, while when $s = 2\ell$, $\ell = 1, 2, \ldots$, the total number of

Hermitian (real symmetric) solutions is given by (23), (25), respectively, since the h.s. and r.s.s. depend on all the permutations of the sign of sth root of eigenvalues of Y, (see formula of X in (11)). □

Remark 3.1.

(i) It is clear that for $s = 1$, the number of solutions in (24) and (26) is the same as in [3, Theorem 9].

(ii) We remind that the Hermitian solutions are linear dependent according to statement (ii) in Remark 2.1; consequently, when s is an even number, the number of linear independent Hermitian solutions of (1) is equal to

$$\# \, h.s. = 2^{n-1} \prod_{j=1}^{m} (n_j + 1), \tag{27}$$

and the number of linear independent real symmetric solutions of (1) is equal to

$$\# \, r.s.s. = 2^{n-1} \prod_{k=1}^{p+q} (n_k + 1). \tag{28}$$

4 Maximal solution of $X^s - A^* X^{-s} A = Q$

It is well known [3] that, when $A \in \mathcal{M}_n$ is nonsingular and $Q > 0$, there always exist the extreme solutions of equation $X - A^* X^{-1} A = Q$, a unique positive definite solution, which is the maximal, and a unique negative definite solution, which is the minimal; in this section the Hermitian solutions of the matrix equation

$$X^s - A^* X^{-s} A = Q \tag{29}$$

are discussed. It will be proved that these solutions are related to the maximal solution of an equation of the type of (5). The matrix equation (29) using the

Matrix Inversion Lemma[4] can be written:

$$\begin{aligned} X^s &= Q + A^* X^{-s} A = Q + A^* \left(X^s\right)^{-1} A = Q + A^* \left(Q + A^* X^{-s} A\right)^{-1} A \\ &= Q + A^* \left[Q^{-1} - Q^{-1} A^* \left(X^s + A Q^{-1} A^*\right)^{-1} A Q^{-1}\right] A \\ &= Q + A^* Q^{-1} A - A^* Q^{-1} A^* \left(X^s + A Q^{-1} A^*\right)^{-1} A Q^{-1} A \end{aligned}$$

Setting in the above equality

$$B = AQ^{-1}A \tag{30}$$

$$C = Q + A^* Q^{-1} A + A Q^{-1} A^* \tag{31}$$

$$Y = X^s + A Q^{-1} A^* \tag{32}$$

we have :

$$Y + B^* Y^{-1} B = C \tag{33}$$

Remind that the equation in (33) is of the type of (5); C in (31) is a positive definite matrix as sum of positive definite matrices (noticing that $A^*Q^{-1}A, AQ^{-1}A^*$ are positive definite matrices), and B in (30) is a nonsingular matrix due to $detB = (detA)^2(detQ)^{-1} \neq 0$.
Thus, it becomes obvious that the solutions of (29) can be derived through the solutions of (33), whose existence is related to the numerical radius of $C^{-1/2}BC^{-1/2}$ [2, 3, 8].

Moreover, rewritting (32) as

$$X^s = Y - AQ^{-1}A^*, \tag{34}$$

it is evident that the solving of (29) is related to the existence of sth root of the matrix $Y - AQ^{-1}A^*$ in (34), the existence of which is guaranteed by the following lemma.

Lemma 4.1. [9, Theorem 7.2.6] *Let $A \in \mathcal{M}_n$ be a positive semidefinite matrix and let $\nu \geq 1$ be a given integer. Then there exists a unique positive semidefinite matrix B such that $B^\nu = A$.*

In the following theorem conditions and formulas for the solutions of (29) are presented.

[4] We remind that for the matrices $K, L, M, N \in \mathcal{M}_n$ with K, M nonsingular the Matrix Inversion Lemma is given: $(K + LMN)^{-1} = K^{-1} - K^{-1}L(M^{-1} + NK^{-1}L)^{-1}NK^{-1}$

Theorem 4.1. *Let $B \in \mathcal{M}_n$ be the nonsingular matrix in (30), $C \in \mathcal{M}_n$ be the positive matrix in (31), with $r(C^{-1/2}BC^{-1/2}) \leq \frac{1}{2}$. Then,*

(i) *there exists a positive definite solution $Y \in \mathcal{M}_n$ of (33).*

(ii) *If $Y_{max} \in \mathcal{M}_n$ is a maximal solution of (33) with*

$$\lambda_{min}(Y_{max}) > \lambda_{max}(AQ^{-1}A^*),$$

then the associated $\Psi \equiv Y_{max} - AQ^{-1}A^*$ by (34) is a positive definite matrix and the unique maximal solution $X_{max} \in \mathcal{M}_n$ of the matrix equation (29) is formed

$$X_{max} = U diag\left(\sqrt[s]{\lambda_1(\Psi)}, \sqrt[s]{\lambda_2(\Psi)}, \ldots, \sqrt[s]{\lambda_n(\Psi)}\right) U^*, \qquad (35)$$

where $\lambda_i(\Psi) \in \sigma(\Psi)$, $i = 1, 2, \ldots, n$, and $U \in \mathcal{M}_n$ is a unitary matrix, which diagonalizes Ψ.

Additionally, if $s = 2\ell$, $\ell = 1, 2, \ldots,$ then there exist 2^n Hermitian solutions $X \in \mathcal{M}_n$ of (29) including the maximal one, which are derived by (35) using 2^n combinations of the signed $\sqrt[s]{\lambda_i(\Psi)}$, with $\lambda_i(\Psi) \in \sigma(\Psi)$.

Proof. **(i)** By the assumption of the numerical radius of $C^{-1/2}BC^{-1/2}$ and Proposition 2.1 for $s = 1$, it is obvious that the matrix equation (33) has at least one positive definite solution.

(ii) Since Y_{max} is a Hermitian matrix, it is clear that Ψ is one. Also, $Y_{max}, AQ^{-1}A^*$ are positive definite matrices, thus its eigenvalues are indexed in increasing order as

$$0 < \lambda_{min}(Y_{max}) \leq \cdots \leq \lambda_{max}(Y_{max}),$$
$$0 < \lambda_{min}(AQ^{-1}A^*) \leq \cdots \leq \lambda_{max}(AQ^{-1}A^*),$$

respectively. According to Weyl's Theorem [9, Theorem 4.3.1] for the form of $\Psi \equiv Y_{max} - AQ^{-1}A^*$ the following inequality holds:

$$\lambda_{min}(Y_{max}) - \lambda_{max}(AQ^{-1}A^*) \leq \lambda_{min}(Y_{max} - AQ^{-1}A^*) \leq \lambda_{min}(Y_{max}) - \lambda_{min}(AQ^{-1}A^*)$$

Combining the assumption $\lambda_{min}(Y_{max}) > \lambda_{max}(AQ^{-1}A^*)$ with the left part of the above inequality we have

$$\lambda_{min}(\Psi) = \lambda_{min}(Y_{max} - AQ^{-1}A^*) \geq \lambda_{min}(Y_{max}) - \lambda_{max}(AQ^{-1}A^*) > 0$$

from which arises $\Psi > 0$. Since the assumption of Lemma 4.1 is satisfied by Ψ, there exists a unique positive definite solution of $X^s = \Psi$, which is denoted $X_{max} \in \mathcal{M}_n$. Using the unitary matrix $U \in \mathcal{M}_n$, which arises from the diagonalization of Ψ, and the formula of X_{max} in (35) the matrix equation $X_{max}^s = \Psi$ directly is verified; more specifically X_{max} is a maximal solution of (29) due to the uniqueness of maximal Y_{max} of (33).

It is evident that, when $s = 2\ell$, $\ell = 1, 2, \dots$, using all the possible combinations of the signed $\sqrt[s]{\lambda_i(\Psi)}$ in the diagonal matrix of the formula of the solutions in (35), all the Hermitian solutions of (29) derived, the total number of which is equal to 2^n. □

Remark 4.1.

(i) Notice that the solution of (29) does not depend only on the solution of (33), but it is based on the definiteness of the matrix $Y - AQ^{-1}A^*$ in (34) as well; thus searching Hermitian or positive definite solutions of (29) it is required $\Psi > 0$, where Ψ is given as in Theorem 4.1.

(ii) For the computation of Y_{max} in (ii) of Theorem 4.1 the associated formula in (22) can be used.

5 Simulation results

Simulation results are given to illustrate the efficiency of the proposed method. The proposed method computes the Hermitian solutions as verified through the following simulation examples, using Matlab 6.5.

Example 5.1. Let the matrices

$$A = \begin{bmatrix} -0.2 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.2 & -0.4 \\ -0.4 & 2.3 \end{bmatrix}$$

in the equation $X^s + A^*X^{-s}A = Q$, $s = 2, 3$. Obviously, A is a nonsingular matrix, with spectrum $\sigma(A) = \{\lambda_1(A) = -0.3179, \ \lambda_2(A) = 0.8179\}$, and $Q > 0$, with $\sigma(Q) = \{1.0699, 2.4301\}$. Since $r(Q^{-1/2}AQ^{-1/2}) = 0.4691$, Proposition 2.1 guarantees the existence of positive definite solutions of (1) and Theorem 2.3 determines the definiteness of these solutions from (10)-(11). The spectrum of matrix Φ^+ in (18) is

$$\begin{aligned}\sigma(\Phi^+) &= \{\lambda_1(\Phi^+) = -19.7683,\ \lambda_2(\Phi^+) = -2.0634,\ \lambda_3(\Phi^+) = -0.0506,\\ &\quad \lambda_4(\Phi^+) = -0.4846\},\end{aligned}$$

with

$$\left|\lambda_1(\Phi^+)\right| = 19.7683,\ \left|\lambda_2(\Phi^+)\right| = 2.0634,\ \left|\lambda_3(\Phi^+)\right| = 0.0506,\ \left|\lambda_4(\Phi^+)\right| = 0.4846,$$

thus Φ^+ has $m = p = 2$ real eigenvalues outside the unit circle, the algebraic multiplicity of which is $n_1 = n_2 = 1$, and $q = 0$ (all eigenvalues are real numbers). According to Theorem 3.1 the Riccati Equation Solution Method can be applied, because its assumptions are verified; the associated number of Hermitian solutions coincides to the real symmetric one and for $s = 2$, $s = 3$ the solutions are computed by (23), (24), respectively, which is equal to

$$\#\, h.s. \equiv r.s.s. = 2^2 \prod_{j=1}^{m}(n_j+1) = 16, \text{ and } \#\, h.p.d.s. \equiv r.p.s.s. = \prod_{j=1}^{m}(n_j+1) = 4.$$

The corresponding eigenvectors of Φ^+ are the columns of the matrix

$$W = \begin{bmatrix} 0.5686 & -0.3944 & 0.9655 & -0.4954 \\ -0.1045 & -0.5421 & -0.2506 & -0.7263 \\ 0.6915 & -0.1982 & 0.0660 & -0.1333 \\ -0.4331 & -0.7150 & -0.0257 & -0.4574 \end{bmatrix}.$$

The positive definite solutions Y_j of (5) are computed by (21);

- when $s = 2$, according to Theorem 2.3 for each Y_j the corresponding Hermitian solutions of (1) are derived by (11) as follows:

$$Y_1 = \begin{bmatrix} 0.0985 & 0.1163 \\ 0.1163 & 0.5504 \end{bmatrix},$$

$$X_1 = \begin{bmatrix} 0.2927 & 0.1134 \\ 0.1134 & 0.7332 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} -0.2084 & 0.2347 \\ 0.2347 & 0.7038 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} -0.2927 & -0.1134 \\ -0.1134 & -0.7332 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 0.2084 & -0.2347 \\ -0.2347 & -0.7038 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 0.1373 & 0.2657 \\ 0.2657 & 1.1258 \end{bmatrix},$$

$$X_5 = \begin{bmatrix} 0.3147 & 0.1957 \\ 0.1957 & 1.0428 \end{bmatrix}, \qquad X_6 = \begin{bmatrix} -0.1845 & 0.3213 \\ 0.3213 & 1.0112 \end{bmatrix},$$

$$X_7 = \begin{bmatrix} -0.3147 & -0.1957 \\ -0.1957 & -1.0428 \end{bmatrix}, \quad X_8 = \begin{bmatrix} 0.1845 & -0.3213 \\ -0.3213 & -1.0112 \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} 1.1106 & -0.5740 \\ -0.5740 & 1.0213 \end{bmatrix},$$

$$X_9 = \begin{bmatrix} 1.0133 & -0.2897 \\ -0.2897 & 0.9682 \end{bmatrix}, \quad X_{10} = \begin{bmatrix} -0.3674 & 0.9877 \\ 0.9877 & -0.2137 \end{bmatrix},$$

$$X_{11} = \begin{bmatrix} -1.0133 & 0.2897 \\ 0.2897 & -0.9682 \end{bmatrix}, \quad X_{12} = \begin{bmatrix} 0.3674 & -0.9877 \\ -0.9877 & 0.2137 \end{bmatrix},$$

$$Y_4 = \begin{bmatrix} 1.1319 & -0.4580 \\ -0.4580 & 1.6523 \end{bmatrix},$$

$$X_{13} = \begin{bmatrix} 1.0454 & -0.1978 \\ -0.1978 & 1.2701 \end{bmatrix}, \quad X_{14} = \begin{bmatrix} -0.3443 & -1.0067 \\ -1.0067 & 0.7993 \end{bmatrix},$$

$$X_{15} = \begin{bmatrix} -1.0454 & 0.1978 \\ 0.1978 & -1.2701 \end{bmatrix}, \quad X_{16} = \begin{bmatrix} 0.3443 & 1.0067 \\ 1.0067 & -0.7993 \end{bmatrix}.$$

Also, we observe that the Hermitian solutions are in pairs opposite, thus it is verified the statement (ii) in Remark 2.1 and the number of linear independent solutions is given by (27)

$$\# h.s. \equiv r.s.s. = 2 \prod_{j=1}^{m} (n_j + 1) = 8,$$

and the associated solutions are $X_1, X_2, X_5, X_6, X_9, X_{10}, X_{13}, X_{14}$.

- when $s = 3$, according to Theorem 2.3 for each Y_j the corresponding positive definite solutions of (1) are derived by (10) as follows:

$$Y_1 = \begin{bmatrix} 0.0985 & 0.1163 \\ 0.1163 & 0.5504 \end{bmatrix}, \quad X_1 = X_{min} = \begin{bmatrix} 0.4361 & 0.0962 \\ 0.0962 & 0.8100 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 0.1373 & 0.2657 \\ 0.2657 & 1.1258 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0.4516 & 0.1533 \\ 0.1533 & 1.0219 \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} 1.1106 & -0.5740 \\ -0.5740 & 1.0213 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0.9993 & -0.1950 \\ -0.1950 & 0.9689 \end{bmatrix},$$

$$Y_4 = \begin{bmatrix} 1.1319 & -0.4580 \\ -0.4580 & 1.6523 \end{bmatrix}, \quad X_4 = X_{max} = \begin{bmatrix} 1.0262 & -0.1260 \\ -0.1260 & 1.1693 \end{bmatrix},$$

It is also verified that: $0 < X_{min} < X_{max}$.

Example 5.2. Consider $s = 3$ in equation $X^s + A^*X^{-s}A = Q$ with

$$A = \begin{bmatrix} 0.32 & 0.13 & 0.12 \\ 0.20 & 0.34 & 0.12 \\ 0.11 & 0.17 & 0.10 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.20 & -0.30 & 0.10 \\ -0.30 & 2.10 & 0.20 \\ 0.10 & 0.20 & 0.65 \end{bmatrix}.$$

In this example, A is a nonsingular matrix with

$$\sigma(A) = \{\lambda_1(A) = 0.5660,\ \lambda_2(A) = 0.1602,\ \lambda_3(A) = 0.0337\},$$

Q is a positive definite matrix with

$$\sigma(Q) = \{\lambda_{min}(Q) = 0.5882,\ \lambda_2(Q) = 1.1538,\ \lambda_{max}(Q) = 2.2080\},$$

and the spectrum of A^*A is

$$\sigma(A^*A) = \{\lambda_{min}(A^*A) = 0.0010,\ \lambda_2(A^*A) = 0.0297,\ \lambda_{max}(A^*A) = 0.3241\}.$$

From the previous values of the spectrum it is obvious that

$$\lambda_{max}(A^*A) > \frac{1}{4}\left(\lambda_{min}(Q)\right)^2,$$

i.e., the inequality in (13) is not satisfied, hence Theorem 3.1 in [6] can not be applied.

Also, the inequality in (14) is verified; the real root of equation $x^6 - \lambda_{max}(Q)x^3 + \lambda_{min}(A^*A) = 0$ is $\alpha_1 = 0.0761$, for which holds $\alpha_1 \in \left(0,\ \sqrt[3]{\frac{1}{2}\lambda_{min}(Q)}\right)$ and yields

$$\lambda_{max}(A^*A) > \frac{\alpha_1^2}{2}\sqrt[3]{\frac{1}{2}\left(\lambda_{min}(Q)\right)^4}.$$

Thus the inequality in (15) is not verified, which means that neither Theorem 2.3 in [4] can be applied.

Moreover, the entries of $Q^{-1/2}AQ^{-1/2}$ are nonnegative real numbers

$$Q^{-1/2}AQ^{-1/2} = \begin{bmatrix} 0.2886 & 0.1118 & 0.1087 \\ 0.1606 & 0.1724 & 0.0758 \\ 0.0965 & 0.1226 & 0.1124 \end{bmatrix},$$

with $\|Q^{-1/2}H_AQ^{-1/2}\| = 0.4392$, i.e., the inequality (8) is satisfied, thus, Proposition 2.2 guarantees the existence of positive definite solutions. Since the spectrum of Φ^+ is $\sigma(\Phi^+) = \{\lambda_{1,2}(\Phi^+) = -204.227 \pm 57.086\mathbf{i},\ \lambda_3(\Phi^+) = 2.835,\ \lambda_4(\Phi^+) = -0.352,\ \lambda_{5,6}(\Phi^+) = -0.0045 \pm 0.0013\mathbf{i}\}$, with $|\lambda_{1,2}(\Phi^+)| = 212.0557$, $|\lambda_3(\Phi^+)| = 2.8351$, $|\lambda_4(\Phi^+)| = 0.3527$, $|\lambda_{5,6}(\Phi^+)| = 0.0047$, Φ^+ has $p = 1$ real eigenvalue and $q = 1$ pair of complex eigenvalues ($m = 3$ in total) outside in unit circle, the algebraic multiplicity of which is equal to 1. According to Theorem 3.1 the Riccati Equation Solution Method can be applied, because its hypotheses are verified and the number of Hermitian solutions is computed by (24) and the number of real symmetric solutions is computed by (26):

$$\#\,h.p.d.s. = \prod_{j=1}^{m}(n_j + 1) = 8, \quad \#\,r.p.s.s. = \prod_{k=1}^{p+q}(n_k + 1) = 4$$

The positive definite solutions are computed by (10) and given in the following:

$$X_1 = X_{min} = \begin{bmatrix} 0.4388 & 0.1899 & 0.1149 \\ 0.1899 & 0.4160 & 0.1630 \\ 0.1149 & 0.1630 & 0.2083 \end{bmatrix}, \quad \text{(minimal solution)}$$

$$X_2 = \begin{bmatrix} 0.5602 & 0.3133 & 0.1862 \\ 0.3133 & 0.5414 & 0.2355 \\ 0.1862 & 0.2355 & 0.2502 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 0.6373 & 0.1286 - 0.0644\mathbf{i} & -0.1748 + 0.0867\mathbf{i} \\ 0.1286 + 0.0644\mathbf{i} & 0.4645 & 0.2128 - 0.1185\mathbf{i} \\ -0.1748 - 0.0867\mathbf{i} & 0.2128 + 0.1185\mathbf{i} & 0.6880 \end{bmatrix},$$

$$X_4 = \begin{bmatrix} 0.6373 & 0.1286 + 0.0644\mathbf{i} & -0.1748 - 0.0867\mathbf{i} \\ 0.1286 - 0.0644\mathbf{i} & 0.4645 & 0.2128 + 0.1185\mathbf{i} \\ -0.1748 + 0.0867\mathbf{i} & 0.2128 - 0.1185\mathbf{i} & 0.6880 \end{bmatrix},$$

$$X_5 = \begin{bmatrix} 0.7760 & 0.2570 - 0.0653\mathbf{i} & -0.1102 + 0.0929\mathbf{i} \\ 0.2570 + 0.0653\mathbf{i} & 0.5835 & 0.2727 - 0.1124\mathbf{i} \\ -0.1102 - 0.0929\mathbf{i} & 0.2727 + 0.1124\mathbf{i} & 0.7186 \end{bmatrix},$$

$$X_6 = \begin{bmatrix} 0.7760 & 0.2570 + 0.0653\mathbf{i} & -0.1102 - 0.0929\mathbf{i} \\ 0.2570 - 0.0653\mathbf{i} & 0.5835 & 0.2727 + 0.1124\mathbf{i} \\ -0.1102 + 0.0929\mathbf{i} & 0.2727 - 0.1124\mathbf{i} & 0.7186 \end{bmatrix},$$

$$X_7 = \begin{bmatrix} 0.8241 & -0.2260 & -0.0688 \\ -0.2260 & 1.1677 & -0.0135 \\ -0.0688 & -0.0135 & 0.8053 \end{bmatrix},$$

$$X_8 = X_{max} = \begin{bmatrix} 0.9947 & -0.1149 & 0.0131 \\ -0.1149 & 1.2415 & 0.0398 \\ 0.0131 & 0.0398 & 0.8447 \end{bmatrix}, \quad \text{(maximal solution)}$$

The real symmetric solutions are X_1, X_2, X_7, X_8 and for every positive definite solution X, it is verified that $0 < X_{min} \leq X \leq X_{max}$.

Example 5.3. Let the matrices A, Q in Example 5.1 and the matrix equation $X^s - A^*X^{-s}A = Q$, for $s = 2, 3$. By (30), (31) the matrices B, C are computed and since the entries of the matrix

$$C^{-1/2}BC^{-1/2} = \begin{bmatrix} 0.0580 & 0.0457 \\ 0.0331 & 0.1354 \end{bmatrix}$$

are nonnegative real numbers, according to Proposition 2.2

$$r(C^{-1/2}BC^{-1/2}) = \|C^{-1/2}H_BC^{-1/2}\| = 0.1521,$$

where H_B is the Hermitian part of B. Thus, according to (i) of Theorem 4.1 the existence of positive definite solutions of (33) is quaranteed, which are computed by (21). Also, the spectrum of matrix Φ^+ in (18) is

$$\begin{aligned}\sigma(\Phi^+) \;\; = \;\; & \{\lambda_1(\Phi^+) = -570.3769,\ \lambda_2(\Phi^+) = -41.4867,\ \lambda_3(\Phi^+) = -0.0241, \\ & \lambda_4(\Phi^+) = -0.0018\},\end{aligned}$$

with

$$\left|\lambda_1(\Phi^+)\right| = 570.3769,\ \left|\lambda_2(\Phi^+)\right| = 41.4867,\ \left|\lambda_3(\Phi^+)\right| = 0.0241,\ \left|\lambda_4(\Phi^+)\right| = 0.0018,$$

thus Φ^+ has $m = p = 2$ real eigenvalues outside the unit circle, the algebraic multiplicity of which is $n_1 = n_2 = 1$, and $q = 0$ (all eigenvalues are real numbers). Since the assumptions of Theorem 3.1 are verified, the Riccati

Equation Solution Method can be applied for the computation of all positive definite solutions Y_j of equation $Y + B^*Y^{-1}B = C$; by (24) the associated number of Y_j is equal to

$$\# \, h.p.d.s. \equiv r.p.s.s. = \prod_{j=1}^{m}(n_j+1) = 4$$

and by (21) the positive definite solutions Y_j are given in the following:

$$Y_1 = \begin{bmatrix} 0.0056 & 0.0127 \\ 0.0127 & 0.0589 \end{bmatrix}, \qquad Y_2 = \begin{bmatrix} 0.1677 & 0.6179 \\ 0.6179 & 2.3193 \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} 1.2599 & -0.7603 \\ -0.7603 & 0.5353 \end{bmatrix}, \quad Y_4 = \begin{bmatrix} 1.3386 & -0.3144 \\ -0.3144 & 3.0618 \end{bmatrix}$$

Also, $Y_4 = Y_{max}$ is the only positive definite solution from the above, which yields

$$\Psi_4 \equiv Y_{max} - AQ^{-1}A^* = \begin{bmatrix} 1.2540 & -0.3875 \\ -0.3875 & 2.6914 \end{bmatrix} > 0$$

due to $\sigma(\Psi_4) = \{1.1561, 2.7892\}$.

- For $s = 2$, according to (ii) of Theorem 4.1 the maximal solution of $X^2 - A^*X^{-2}A = Q$ is computed by (35) and is equal to

$$X_{max} = \begin{bmatrix} 1.1109 & -0.1412 \\ -0.1412 & 1.6345 \end{bmatrix}.$$

There exist others three different Hermitian solutions, which are given in the following:

$$X_1 = \begin{bmatrix} -0.9108 & -0.6515 \\ -0.6515 & 1.5057 \end{bmatrix}, \; X_2 = \begin{bmatrix} -1.1109 & 0.1412 \\ 0.1412 & -1.6345 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 0.9108 & 0.6515 \\ 0.6515 & -1.5057 \end{bmatrix}$$

- For $s = 3$, the positive definite solution of $X^3 - A^*X^{-3}A = Q$ is given in the following, which is computed by (35):

$$X = \begin{bmatrix} 1.0710 & -0.0850 \\ -0.0850 & 1.3862 \end{bmatrix}$$

References

[1] M. Adam, On the positive definite solutions of the matrix equation $X^s + A^*X^{-s}A = Q$, *The Open Applied Mathematics Journal*, **5**, (2011), 19-25.

[2] M. Adam, N. Assimakis and F. Sanida, Algebraic Solutions of the Matrix Equations $X + A^T X^{-1}A = Q$ and $X - A^T X^{-1}A = Q$, *International Journal of Algebra*, **2**(11), (2008), 501-518.

[3] M. Adam, N. Assimakis, G. Tziallas and F. Sanida, Riccati Equation Solution Method for the Computation of the Solutions of $X + A^T X^{-1}A = Q$ and $X - A^T X^{-1}A = Q$, *The Open Applied Informatics Journal*, **3**, (2009), 22-33.

[4] J. Cai and G. Chen, Some investigation on Hermitian positive definite solutions of the matrix equation $X^s + A^*X^{-t}A = Q$ *Linear Algebra Appl.*, **430**, (2009), 2448-2456.

[5] J. Cai and G. Chen, On the Hermitian positive definite solutions of non-linear matrix equation $X^s + A^*X^{-t}A = Q$, *Applied Mathematics and Computation*, **217**, (2010), 117-123.

[6] X. Duan and A. Liao, On the existence of Hermitian positive definite solutions of the matrix equation $X^s + A^*X^{-t}A = Q$, *Linear Algebra Appl.*, **429**, (2008), 673-687.

[7] J.C. Engwerda, On the existence of a positive definite solution of the matrix equation $X + A^T X^{-1}A = I$, *Linear Algebra Appl.*, **194**, (1993), 91-108.

[8] J.C. Engwerda, A.C.M. Ran and A.L. Rijkeboer, Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A^*X^{-1}A = Q$, *Linear Algebra Appl.,* **186**, (1993), 255-275.

[9] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.

[10] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[11] X.G. Liu and H. Gao, On the positive definite solutions of the matrix equations $X^s \pm A^T X^{-t} A = I_n$, *Linear Algebra Appl.*, **368**, (2003), 83-97.

[12] B. Meini, Efficient computation of the extreme solutions of $X + A^* X^{-1} A = Q$ and $X - A^* X^{-1} A = Q$, *Mathematics of Computation*, **71**(239), (2001), 1189-1204.