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ON THE SOLUTION OF THE QUASI RICCATI AND LYAPUNOV EQUATIONS

M. ADAM[1], N. ASSIMAKIS[2], E. FAZAELI[3] AND G. TZIALLAS[2*]

[1]Department of Computer Science and Biomedical Informatics, University of Thessaly, 2-4 Papasiopoulou Str., P.O. 35131 Lamia, Greece.
[2]Department of Electronic Engineering, Technological Educational Institute of Central Greece, 3rd km Old National Road Lamia-Athens, P.O. 35131 Lamia, Greece.
[3]Department of Electrical Engineering, Islamic Azad University, Najafabad Branch, Isfahan, Iran.

AUTHORS' CONTRIBUTIONS

This work was carried out in collaboration between all authors. Author MA designed the study and carried out the proof of proposed method. Author NA designed the methodology and carried out the proof of proposed method. Author EF tested the method, selected the simulation examples and interpreted the results. Author GT designed the simulation and wrote software programs. All authors read and approved the final manuscript.

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ABSTRACT

The classical Riccati equation arises in optimal linear estimation where the state and measurement noise covariance matrices are non-negative definite. The quasi Riccati equation is defined preserving the form of the classical Riccati equation and using noise matrices that are not necessarily non-negative definite. The classical Lyapunov equation results from the classical Riccati equation in the infinite measurement noise case. The quasi Lyapunov equation is defined preserving the form of the classical Lyapunov equation and using complex transition matrix. A method for computing the solution of the quasi Riccati and Lyapunov equations is proposed. The method is based on the algebraic solution of the discrete time Riccati equation, where the eigenvalues of the resultant symplectic matrix are allowed to lie on the unit circle.

Keywords: Riccati equation; Lyapunov equation; algebraic solution.

1 Introduction

In the design of optimal state estimators for linear discrete-time systems, it is necessary to obtain the observer gain (Kalman filter gain) matrix, which is derived from the solution of the discrete-time Riccati equation. In fact, the solution of the Riccati equation has significant role in estimation theory. The classical discrete time Riccati equation that arises in optimal linear estimation and is associated with time invariant systems [1,2] is in the following form:

$$P = Q + FPF^T - FPH^T \left[HPH^T + R \right]^{-1} HPF^T \tag{1}$$

**Corresponding author: Email: gt@teilam.gr;*

where F is the state transition matrix of dimension $n \times n$, H is the output matrix of dimension $m \times n$, Q is the state noise covariance matrix of dimension $n \times n$, R is the measurement noise covariance of dimension $m \times m$. Moreover Q and R are symmetric and non-negative definite. Obviously, P is symmetric. Note that F^T denotes the transpose of F. Equation (1) is sometimes termed as steady state Riccati equation [2]. In view of the importance of the Riccati equation, there exists a considerable literature on the algorithmic aspects of its solution. Iterative algorithms are presented in [2-5]. Distributed algorithms are presented in [6]. Algebraic algorithms are presented in [1,2,7].

The classical Riccati equation requires the matrices Q and R to be non-negative definite. The quasi Riccati equation is defined preserving the form of the classical Riccati equation, using matrices that are not necessarily non-negative definite, deriving a generalization of the classical discrete time Riccati equation.

In the infinite measurement noise case, where $R = \infty$, the classical discrete time Lyapunov equation is derived from the classical discrete time Riccati equation and is in the following form:

$$P = Q + FPF^T \tag{2}$$

The quasi Lyapunov equation is defined preserving the form of the classical Lyapunov equation, using complex transition matrix, deriving a generalization of the classical discrete time Lyapunov equation.

The central issue of this paper is to compute the solution of the algebraic matrix quasi Riccati and Lyapunov equations. In this paper an algebraic method for computing the solution of the matrix quasi Riccati and Lyapunov equations is proposed. The quasi Riccati equation allows the matrices Q and R in (1) to be non-positive definite. The quasi Lyapunov equation allows the matrix F in (2) to be complex. The novelty of the paper is that the eigenvalues of the resultant symplectic matrix are allowed to lie on the unit circle.

The paper is organized as follows: In section 2 the symplectic matrix related to the Riccati and Lyapunov equations is presented. In section 3 the algebraic method for the solution of the quasi Riccati and Lyapunov equations is derived. In section 4 simulation results are given to illustrate the efficiency of the proposed method. Section 5 summarizes the conclusions.

2 Symplectic Matrix Related to Riccati and Lyapunov Equations

By assuming P as a non-singular matrix and R as non-singular and with applying the matrix inversion lemma in the Riccati equation (1), the matrix quasi Riccati equation can be written as follows:

$$P = Q + F\left[P^{-1} + H^T R^{-1} H\right]^{-1} F^T \tag{3}$$

Then, the corresponding iterative form is:

$$P_{k+1} = Q + F\left[P_k^{-1} + H^T R^{-1} H\right]^{-1} F^T \tag{4}$$

Then P_k tends to the solution of (4) as k tends to infinity.

Notice that the convergence properties of (4) are analytically described in [2].

Let define:

$$Y_k = P_k X_k \tag{5}$$

Then

$$\begin{bmatrix} X_{k+1} \\ Y_{k+1} \end{bmatrix} = \Phi \begin{bmatrix} X_k \\ Y_k \end{bmatrix} \tag{6}$$

where

$$\Phi = \begin{bmatrix} F^{-T} & F^{-T}H^T R^{-1} H \\ QF^{-T} & F + QF^{-T}H^T R^{-1} H \end{bmatrix} \tag{7}$$

is a symplectic matrix of dimension $2n \times 2n$.

Notice that the non-singularity of F is necessary for the construction of Φ.

An algebraic nonrecursive method for solving the discrete Riccati equation (1) is presented in [7]. This method is based on the symplectic matrix Φ.

We are going to develop an algebraic method for computing solution of the quasi Riccati equation in section 3. The method uses the ideas of the algebraic solution proposed in [1,2,7]. The method considers the case where all the eigenvalues of Φ are non-zero and are allowed to lie on the unit circle.

In order to show the necessity to allow the matrices Q and R to be non-positive definite, consider the scalar ($n = m = 1$) quasi Riccati equation with real parameters, where the associated symplectic matrix is:

$$\Phi = \begin{bmatrix} \frac{1}{F} & \frac{H^2}{F}\frac{1}{R} \\ \frac{Q}{F} & F + \frac{H^2}{R}\frac{Q}{F} \end{bmatrix}$$

Then, the eigenvalues of Φ satisfy

$$\lambda^2 - \lambda \frac{F^2 + 1 + Q\frac{H^2}{R}}{F} + 1 = 0$$

Note that the eigenvalues of matrix Φ occur in reciprocal pairs, due to the fact that Φ is a symplectic matrix. Note also that $\lambda_1 \lambda_2 = 1$. Consider the case where the eigenvalues of Φ lie on the unit circle. It is clear that if $\lambda_1 = 1$, then also $\lambda_2 = 1$. In this case, we have:

$$\frac{F^2 + 1 + Q\frac{H^2}{R}}{F} = 2 \Rightarrow F^2 - 2F + 1 + Q\frac{H^2}{R} = 0 \Rightarrow (F-1)^2 + Q\frac{H^2}{R} = 0$$

from where it is clear that $QR \le 0$ is required.

In the infinite measurement noise case, where $R = \infty$, the classical discrete time Lyapunov equation is derived from the classical discrete time Riccati equation. Then the corresponding symplectic matrix of dimension $2n \times 2n$ becomes:

$$\Phi = \begin{bmatrix} F^{-T} & 0 \\ QF^{-T} & F \end{bmatrix} \tag{8}$$

The algebraic method for computing solution of the quasi Riccati equation can be used for the solution of the quasi Lyapunov equation. The method considers the case where all the eigenvalues of Φ are non-zero and are allowed to lie on the unit circle.

In order to show the necessity to allow the matrix F to be complex, consider the scalar ($n = m = 1$) quasi Lyapunov equation where the associated symplectic matrix is:

$$\Phi = \begin{bmatrix} \frac{1}{F} & 0 \\ \frac{Q}{F} & F \end{bmatrix}$$

Then, the eigenvalues of Φ satisfy

$$\left(\lambda - \frac{1}{F}\right)(\lambda - F) = 0$$

Consider the case where the eigenvalues of Φ lie on the unit circle. It is then clear that it is required F to be complex. Note that if $F = \pm 1$ the quasi Lyapunov equation becomes $P = Q + P$.

3 Algebraic Solution of the Quasi Riccati and Lyapunov Equations

We are going to propose an algebraic method for computing solution of the quasi Riccati equation and quasi Lyapunov equation. The classical algebraic nonrecursive method for solving the classical discrete Riccati equation is presented in [7] and referred in [2]. This method is based on the symplectic matrix Φ. The algorithm we are going to develop uses the ideas of the classical algorithm and considers the case where all the eigenvalues of Φ are non-zero and are allowed to lie on the unit circle.

The symplectic matrix Φ in (7) and (8) can be written as:

$$\Phi = W \ell W^{-1} \tag{9}$$

ℓ is the matrix which contains the eigenvalues of matrix Φ :

$$\ell = \begin{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \Lambda_2^{-1} & 0 \\ 0 & \Lambda_1^{-1} \end{bmatrix} \end{bmatrix} \tag{10}$$

with Λ_1 the diagonal matrix which contains the eigenvalues of matrix Φ that lie outside the unit circle and Λ_2 the diagonal matrix which contains the half of the eigenvalues of matrix Φ that lie on the unit circle. Note that the eigenvalues of matrix Φ occur in reciprocal pairs, due to the fact that Φ is a symplectic matrix. Assume that all the eigenvalues are distinct, except of $1, -1$ which are double, if there exist.

Let W is the matrix which contains the eigenvectors of matrix Φ :

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{11}$$

Then we have:

$$\begin{bmatrix} X_k \\ Y_k \end{bmatrix} = \Phi^k \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} \quad (12)$$

Let now define:

$$\begin{bmatrix} X_k \\ Y_k \end{bmatrix} = W \begin{bmatrix} U_k \\ S_k \end{bmatrix} \quad (13)$$

Then, from (9), (12) and (13), we have:

$$\begin{bmatrix} U_k \\ S_k \end{bmatrix} = \ell^k \begin{bmatrix} U_0 \\ S_0 \end{bmatrix} \quad (14)$$

Then, denoting:

$$U_k = \begin{bmatrix} U_k^u \\ U_k^d \end{bmatrix}$$

and

$$S_k = \begin{bmatrix} S_k^u \\ S_k^d \end{bmatrix}$$

(where u and d for up and down, respectively)

we derive:

$$\begin{aligned} U_k^u &= \Lambda_1^k U_0^u \\ U_k^d &= \Lambda_2^k U_0^d \\ S_k^u &= \Lambda_2^{-k} S_0^u \\ S_k^d &= \Lambda_2^{-k} S_0^d \end{aligned}$$

Furthermore, denoting:

$$X_k = \begin{bmatrix} X_k^u \\ X_k^d \end{bmatrix}$$

and

$$Y_k = \begin{bmatrix} Y_k^u \\ Y_k^d \end{bmatrix}$$

(where u and d for up and down, respectively) and

$$W = \begin{bmatrix} W_{11}^{ul} & W_{11}^{ur} & W_{12}^{ul} & W_{12}^{ur} \\ W_{11}^{dl} & W_{11}^{dr} & W_{12}^{dl} & W_{12}^{dr} \\ W_{21}^{ul} & W_{21}^{ur} & W_{22}^{ul} & W_{22}^{ur} \\ W_{21}^{dl} & W_{21}^{dr} & W_{22}^{dl} & W_{22}^{dr} \end{bmatrix} \tag{15}$$

(where l and r for left and right, respectively)

we derive:

$$\begin{bmatrix} X_k^u \\ X_k^d \\ Y_k^u \\ Y_k^d \end{bmatrix} = \begin{bmatrix} W_{11}^{ul} U_k^u + W_{11}^{ur} U_k^d + W_{12}^{ul} S_k^u + W_{12}^{ur} S_k^d \\ W_{11}^{ul} U_k^u + W_{11}^{ur} U_k^d + W_{12}^{ul} S_k^u + W_{12}^{ur} S_k^d \\ W_{21}^{ul} U_k^u + W_{21}^{ur} U_k^d + W_{22}^{ul} S_k^u + W_{22}^{ur} S_k^d \\ W_{21}^{dl} U_k^u + W_{21}^{dr} U_k^d + W_{22}^{dl} S_k^u + W_{22}^{dr} S_k^d \end{bmatrix}$$

When k tends to infinity, we have:

$$S_k^d = \Lambda_1^{-k} S_0^d \to 0$$

Then we have:

$$\begin{bmatrix} X_\infty^u \\ X_\infty^d \\ Y_\infty^u \\ Y_\infty^d \end{bmatrix} = \begin{bmatrix} W_{11}^{ul} U_\infty^u + W_{11}^{ur} U_\infty^d + W_{12}^{ul} S_\infty^u \\ W_{11}^{ul} U_\infty^u + W_{11}^{ur} U_\infty^d + W_{12}^{ul} S_\infty^u \\ W_{21}^{ul} U_\infty^u + W_{21}^{ur} U_\infty^d + W_{22}^{ul} S_\infty^u \\ W_{21}^{dl} U_\infty^u + W_{21}^{dr} U_\infty^d + W_{22}^{dl} S_\infty^u \end{bmatrix}$$

Then from (15) we have:

$$\begin{bmatrix} X_\infty^u \\ X_\infty^d \end{bmatrix} = \begin{bmatrix} W_{11}^{ul} & W_{11}^{ur} & W_{12}^{ul} \\ W_{11}^{dl} & W_{11}^{dr} & W_{12}^{dl} \end{bmatrix} \begin{bmatrix} U_\infty^u \\ U_\infty^d \\ S_\infty^u \end{bmatrix}$$

and

$$\begin{bmatrix} Y_\infty^u \\ Y_\infty^d \end{bmatrix} = \begin{bmatrix} W_{21}^{ul} & W_{21}^{ur} & W_{22}^{ul} \\ W_{21}^{dl} & W_{21}^{dr} & W_{22}^{dl} \end{bmatrix} \begin{bmatrix} U_\infty^u \\ U_\infty^d \\ S_\infty^u \end{bmatrix}$$

Then from (5) we have:

$$\begin{bmatrix} W_{21}^{ul} & W_{21}^{ur} \\ W_{21}^{dl} & W_{21}^{dr} \end{bmatrix} \begin{bmatrix} U_\infty^u \\ U_\infty^d \end{bmatrix} + \begin{bmatrix} W_{22}^{ul} \\ W_{22}^{dl} \end{bmatrix} S_\infty^u = P_\infty \begin{bmatrix} W_{11}^{ul} & W_{11}^{ur} \\ W_{11}^{dl} & W_{11}^{dr} \end{bmatrix} + P_\infty \begin{bmatrix} W_{12}^{ul} \\ W_{12}^{dl} \end{bmatrix} S_\infty^u$$

Finally, as k tends to infinity P_k tends to the solution P of the quasi Riccati equation (3) or of the quasi Lyapunov equation (2):

$$\begin{bmatrix} W_{21}^{ul} & W_{21}^{ur} \\ W_{21}^{dl} & W_{21}^{dr} \end{bmatrix} = P \begin{bmatrix} W_{11}^{ul} & W_{11}^{ur} \\ W_{11}^{dl} & W_{11}^{dr} \end{bmatrix} \tag{16}$$

and

$$\begin{bmatrix} W_{22}^{ul} \\ W_{22}^{dl} \end{bmatrix} S_{\infty}^{u} = P \begin{bmatrix} W_{12}^{ul} \\ W_{12}^{dl} \end{bmatrix} S_{\infty}^{u} \tag{17}$$

In fact S_{∞}^{u} in (17) is not known; thus we are not able to compute P by (17). Then we are going to compute P by (16). From (16) it is obvious that, if the block matrix W_{11} is non-singular, then

$$P = W_{21} W_{11}^{-1} \tag{18}$$

satisfies the quasi Riccati equation or the quasi Lyapunov equation.

Note that the proposed algorithm for solving the quasi Riccati equation is a generalization of the classical algorithm described in [7] for solving the classical Riccati equation. In fact, in [7] there were no eigenvalues of Φ on the unit circle. Here, the eigenvalues of Φ are allowed to lie on the unit circle. The proposed algorithm has the same structure as the classical algorithm: the proposed algorithm is based on the eigenvalues and the eigenvectors of the symplectic matrix Φ.

In the case where all the eigenvalues of the symplectic matrix Φ in (7) and (8) are distinct and lie on the unit circle (there exist no eigenvalues outside or inside the unite circle), we observe that the solutions in (18) of the quasi Riccati and Lyapunov equations do not depend on the permutation of the first n columns of W, which are eigenvectors of Φ. Then, the solutions calculated by (18) can be derived from any arrangement of the first n eigenvectors of Φ, which has $2n$ eigenvectors. Then, working as in [8], the finite number of the possible permutations and hence the number of solutions of the quasi Riccati and Lyapunov equations is:

$$s = \binom{2n}{n} = \frac{(2n)!}{(n!)^2} \tag{19}$$

4 Simulation Results

Simulation results are given to illustrate the efficiency of the proposed method. The proposed method computes accurate solutions as verified trough the following simulation examples.

Example 1.

Consider the scalar ($n = m = 1$) quasi Riccati equation with

$$F = \frac{\sqrt{2}}{2}, \; H = 1, \; Q = -1, \; R = 2 .$$

The symplectic matrix Φ in (7) is

$$\Phi = \begin{bmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} \\ -\sqrt{2} & 0 \end{bmatrix}$$

and its eigenvalues $\lambda_{1,2} = \frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2}$ lie on the unit circle, since $\left|\lambda_{1,2}\right| = 1$.

The matrix W which contains the eigenvectors of matrix Φ is:

$$W = \begin{bmatrix} -0.4082 - 0.4082i & -0.4082 + 0.4082i \\ 0.8165 & 0.8165 \end{bmatrix}$$

Thus

$$P = W_{21}W_{11}^{-1} = -1 + j$$

satisfies the quasi Riccati equation.

Due to the fact that all the eigenvalues of the symplectic matrix Φ are distinct and lie on the unit circle (there exist no eigenvalues outside or inside the unite circle), there exist $s = \binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \frac{(2)!}{(1!)^2} = 2$ solutions of the quasi Riccati equation. The second solution is:

$$P = W_{22}W_{12}^{-1} = -1 - j$$

Example 2.

Consider the matrix quasi Riccati equation with

$$F = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},\ H = \begin{bmatrix} 1 & 0 \end{bmatrix},\ Q = \begin{bmatrix} 0.3 & 0 \\ 0 & 0 \end{bmatrix},\ R = 0.1$$

This model is derived from a first order stochastic Non-Minimal State Space (NMSS) model for describing a Proportional-Integral-Plus Linear Quadratic Gaussian (PIP-LQG) control problem, as proposed in [9].

The symplectic matrix Φ in (7) is

$$\Phi = \begin{bmatrix} 1 & 1 & 10 & 0 \\ 0 & 1 & 0 & 0 \\ 0.3 & 0.3 & 4 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

and its eigenvalues are

$$\lambda_1 = 4.7913$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 0.2087$$

The matrix W which contains the eigenvectors of matrix Φ is:

$$W = \begin{bmatrix} -0.9310 & 0 & 0 & -0.9920 \\ 0 & 2\times10^{-15} & 0 & 0 \\ -0.3530 & 0 & 0 & 0.0785 \\ 0.0931 & -1 & 1 & 0.0992 \end{bmatrix}$$

Thus

$$P = W_{21}W_{11}^{-1} = \begin{bmatrix} 0.4 & -0.1 \\ -0.1 & -4.5036\times10^{14} \end{bmatrix}$$

satisfies the quasi Riccati equation.

Example 3.

Consider the matrix quasi Riccati equation with

$$F = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -3 \\ 3 & -2 & -6 \end{bmatrix},\ H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ R = -\begin{bmatrix} 11 & 4 & 3 \\ 4 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

The symplectic matrix Φ in (6) is

$$\Phi = \begin{bmatrix} -6 & -3 & -2 & -1 & 2 & 3 \\ -2 & 0 & -1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 0 & 0 & -1 \\ -6 & -3 & -2 & 0 & 1 & 1 \\ -2 & 0 & -1 & 1 & -1 & -1 \\ 3 & 1 & 1 & 3 & -2 & -7 \end{bmatrix}$$

and its eigenvalues are

$$\lambda_1 = -10.9993$$
$$\lambda_2 = 0.0451 + 0.9990i$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$
$$\lambda_5 = 0.0451 - 0.9990i$$
$$\lambda_6 = -0.0909$$

Note that λ_1 lies outside the unit circle, $\lambda_2, \lambda_3, \lambda_4, \lambda_5$ lie on the unit circle and λ_6 lies in the unit circle. The matrix W which contains the eigenvectors of matrix Φ is:

$$W = \begin{bmatrix} -0.5386 & 0.2286+0.0231i & -0.3536 & -0.3536 & 0.2286-0.0231i & -0.2527 \\ -0.2034 & -0.7397 & 0.0000 & 0.0000 & -0.7397 & 0.2453 \\ 0.2095 & 0.2286-0.0231i & 0.3536 & 0.3536 & 0.2286+0.0231i & 0.6497 \\ -0.3762 & 0.0873-0.1581i & -0.7071 & -0.7071 & 0.0873+0.1581i & 0.5854 \\ 0.0203 & -0.3699+0.3536i & -0.3536 & -0.3536 & -0.3699-0.3536i & 0.2699 \\ 0.6948 & 0.1413-0.1812i & -0.3536 & -0.3536 & 0.1413+0.1812i & 0.1959 \end{bmatrix}$$

Thus

$$P = W_{21}W_{11}^{-1} = \begin{bmatrix} 0.2350-0.1235i & -0.5908+0.1998i & -1.7650-0.1235i \\ -0.5908+0.1998i & -0.1742-0.3233i & -1.5908+0.1998i \\ -1.7650-0.1235i & -1.5908+0.1998i & -2.7650-0.1235i \end{bmatrix}$$

satisfies the quasi Riccati equation.

Example 4.

Consider the matrix Riccati equation with

$$F = \begin{bmatrix} -0.02 & -1.4 & 9.8 \\ -0.01 & -0.4 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ Q = \begin{bmatrix} 9.8 & 0 & 0 \\ 0 & 6.3 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ R = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

This model is proposed in [10] for describing a helicopter near hover.

The symplectic matrix Φ in (6) is

$$\Phi = \begin{bmatrix} 0.8333 & 2.5000 & 3.3333 & 3.3333 \\ -5.8333 & -7.5000 & -13.3333 & -13.3333 \\ 0.8333 & 2.5000 & 2.4333 & 4.0333 \\ -17.5000 & -22.5000 & -40.3000 & -39.9000 \end{bmatrix}$$

and its eigenvalues are

$$\lambda_1 = -101.5511$$
$$\lambda_2 = 76.5749$$
$$\lambda_3 = 24.5735$$
$$\lambda_4 = 0.0407$$
$$\lambda_5 = 0.0131$$
$$\lambda_6 = -0.0098$$

The matrix W which contains the eigenvectors of matrix Φ is:

$$W = \begin{bmatrix} -0.0001 & -0.0004 & 0.0033 & 0.7727 & 0.9627 & 0.9769 \\ -0.1560 & -0.1546 & 0.1436 & -0.0053 & -0.0126 & 0.0163 \\ 0.0361 & -0.1910 & 0.4303 & 0.6292 & 0.2524 & -0.1893 \\ -0.0159 & 0.0119 & -0.0040 & -0.0773 & -0.0963 & -0.0977 \\ -0.9869 & -0.9692 & 0.8904 & -0.0013 & -0.0002 & -0.0001 \\ 0.0097 & -0.0127 & 0.0362 & -0.0321 & -0.0129 & 0.0095 \end{bmatrix}$$

Thus

$$P = W_{21}W_{11}^{-1} = \begin{bmatrix} 14.7272 & 0.0564 & -0.1408 \\ 0.0564 & 6.3158 & -0.0394 \\ -0.1408 & -0.0394 & 0.0984 \end{bmatrix}$$

satisfies the Riccati equation.

This example shows that the proposed method for solving the quasi Riccati equation can be used to solve the classical Riccati equation: The proposed algorithm for solving the quasi Riccati and Lyapunov equations is a generalization of the classical algorithm for solving the classical Riccati and Lyapunov equations. Concerning the comparison of the proposed and the classical algorithms, the proposed and the classical algorithms have the same structure: they compute the solution through the eigenvectors of the associated symplectic matrix.

Example 5.

Consider the scalar ($n = m = 1$) quasi Lyapunov equation with

$$F = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2},\ Q = 1\ .$$

The symplectic matrix Φ in (8) is

$$\Phi = \begin{bmatrix} \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \end{bmatrix}$$

and its eigenvalues $\lambda_{1,2} = \frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2}$ lie on the unit circle, since $\left|\lambda_{1,2}\right| = 1$.

The matrix W which contains the eigenvectors of matrix Φ is:

$$W = \begin{bmatrix} 0.8165 & 0 \\ 0.4082 + 0.4082\mathrm{i} & 1 \end{bmatrix}$$

Thus

$$P = W_{21}W_{11}^{-1} = \frac{1}{2}(1 + j)$$

satisfies the quasi Lyapunov equation.

5 Conclusions

The classical Riccati and Lyapunov equations arise in linear estimation. The quasi Riccati equation is defined preserving the form of the classical Riccati equation and using noise matrices that are not necessarily non-negative definite. The quasi Lyapunov equation is defined preserving the form of the classical

Lyapunov equation and using complex transition matrix. The quasi Riccati and Lyapunov equations are generalizations of the classical Riccati and Lyapunov equations.

A general algebraic method for computing the solution of the quasi Riccati and Lyapunov equations is proposed. The proposed algorithm for solving the quasi Riccati and Lyapunov equations is a generalization of the classical algorithm for solving the classical Riccati and Lyapunov equations. The proposed algorithm has the same structure as the classical algorithm: The proposed algorithm is based on the eigenvalues (that are allowed to lie on the unit circle) and the eigenvectors of the associated symplectic matrix. The proposed method provides simple formulas for computing the accurate solution of the quasi Riccati and Lyapunov equations.

Competing Interests

Authors have declared that no competing interests exist.

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