Linear Algebra and its Applications 667 (2023) 165–191

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Sequences of lower and upper bounds for the spectral radius of a nonnegative matrix

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A R T I C L E I N F O

Article history:
Received 16 August 2022
Accepted 6 March 2023
Available online 13 March 2023
Submitted by R. Brualdi

MSC:
15A18
15A42

Keywords:
Nonnegative matrix
Irreducible matrix
Perron-Frobenius theory
Spectral radius
Average $(k+1)$-row sum

A B S T R A C T

In this article, we expand a classical Frobenius' result upon consecutive k-th powers of nonnegative matrices to establish sequences of new lower and upper bounds for the spectral radius with respect to the positive integer k, each term of which is formulated by the average $(k+1)$-row sums of the nonnegative matrix. With the aid of the average $(k+1)$-row sums and taking the extreme entries of the matrix, we study new bounds generalizing existing formulae and we produce sequences of new tighter approximations for the spectral radius. The monotonicity and convergence properties of the constructed sequences are explored and certain conditions are stated under which the new bounds are sharper than Frobenius' bounds and other existing formulae. We further characterize the cases of equality in the aforesaid bounds, when the matrix is irreducible. Throughout, we perform illustrative numerical examples to showcase the efficiency of our proposed bounds and make comparisons among them.
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https://doi.org/10.1016/j.laa.2023.03.006
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1. Introduction

The spectral radius, that is, the largest in magnitude eigenvalue, of a nonnegative matrix plays a central role in many disciplines of applied mathematics and computer science such as graph theory, control theory, cryptography and even biology and epidemic modeling. Indicatively, the stability of a linear invariant discrete-time nonnegative system is ensured whenever the value of the spectral radius of its system matrix does not exceed one (see, [4,5,21] and the references cited therein). Furthermore, the spectral radius of the adjacency matrix of an infectious disease system is the key threshold to control its behavior, since the smaller the spectral radius, the higher the rate at which the disease is eradicated through the network, as opposed to becoming an epidemic. An upper bound for the spectral radius gives a lower bound for the epidemic threshold and thus, if the effective spreading rate is below this lower bound, a safety region is determined in which the virus contamination is guaranteed to die out. The sharper the upper bound for the spectral radius, the less effort is spent to reduce the spreading rate below the lower bound (see, [6,13,19] and the references therein).

Let $\mathcal{M}_n(\mathbb{R})$ be the algebra of $n \times n$ real matrices. The *spectral radius* of $A \in \mathcal{M}_n(\mathbb{R})$ is defined by the set

$$\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ denotes the set of eigenvalues of A, [12]. We refer to $A = [a_{ij}]_{i,j=1}^{n} \in \mathcal{M}_n(\mathbb{R})$ as *nonnegative*, $A \geq 0$ or *positive*, $A > 0$ accordingly, if it is entrywise nonnegative or positive respectively. We recall that a nonnegative matrix $A \in \mathcal{M}_n(\mathbb{R})$ is called *irreducible* if and only if $(I_n + A)^{n-1} > 0$, where I_n denotes the $n \times n$ identity matrix.

The Perron–Frobenius theory concerns the existence of positive or nonnegative eigenvalues and eigenvectors of positive or nonnegative matrices [12,20]. Most notably, it is well-known that when $A \geq 0$ is irreducible, $\rho(A)$ is a positive and simple eigenvalue having a positive eigenvector. Beside to this, Frobenius also proved upper and lower bounds for the spectral radius of a nonnegative matrix involving its row sums, [10]. These results and other extensions have been widely investigated by other researchers in [1–4,8,9,15–18,24] and applied to several problems including stochastic processes [22], Markov chains [23], population models [14], and asynchronous parallel iterative methods [11], among others.

In this work, we contribute new sharper lower and upper bounds for the spectral radius of a nonnegative matrix expanding upon Frobenius' bounds. Towards this, considering the i-th row sum $r_i(A) = \sum_{j=1}^{n} a_{ij}$ of $A \geq 0$ to be a positive quantity for every $i = 1, \dots, n$, we define the i-th *average* $(k+1)$*-row sum* of $A \geq 0$ for a positive integer $k \geq 1$, as the ratio

$$w_i^{(k+1)}(A) = \frac{1}{r_i(A)} \underbrace{\sum_{j=1}^{n} a_{ij} \sum_{u=1}^{n} a_{ju} \cdots \sum_{p=1}^{n} a_{vp}}_{k \text{ times}} r_p(A) = \frac{1}{r_i(A)} \sum_{\tau=1}^{n} a_{i\tau}^{(k)} r_\tau(A), \tag{1.1}$$

where $a_{i\tau}^{(k)}$ denotes the (i,τ)-th element of $A^k \in \mathcal{M}_n(\mathbb{R})$. Note that the latter definition generalizes the associated quantities studied in [24,16,2] for $k = 1, 2, 3$, respectively, developing lower and upper bounds for $\rho(A)$, while in [9] the corresponding bounds have been constructed using only the i-th row sum of A.

We are also interested in the largest diagonal and off-diagonal element of $A \geq 0$,

$$\mu = \max_{1\leq i\leq n}\{a_{ii}\} \text{ and } \nu = \max_{\substack{1\leq i,j\leq n\\ i\neq j}}\{a_{ij}\}, \tag{1.2}$$

respectively, as well as its smallest diagonal and off-diagonal element,

$$s = \min_{1\leq i\leq n}\{a_{ii}\} \text{ and } \tau = \min_{\substack{1\leq i,j\leq n\\ i\neq j}}\{a_{ij}\}, \tag{1.3}$$

respectively. Our approach also involves the positive quantities

$$q = \min_{1\leq i,j\leq n}\left\{\frac{r_j(A)}{r_i(A)} : r_i(A) > 0\right\} \text{ and } b = \max_{1\leq i,j\leq n}\left\{\frac{r_j(A)}{r_i(A)} : r_i(A) > 0\right\}, \tag{1.4}$$

which are interrelated, since

$$q = \frac{\min_{1\leq i,j\leq n} r_j(A)}{\max_{1\leq i,j\leq n} r_i(A)} = \frac{1}{\frac{\max_{1\leq i,j\leq n} r_j(A)}{\min_{1\leq i,j\leq n} r_i(A)}} = \frac{1}{\max_{1\leq i,j\leq n}\left\{\frac{r_j(A)}{r_i(A)}\right\}} = \frac{1}{b}.$$

The article is organized as follows: In Section 2, we generalize a classical Frobenius' result to the k-th powers of a nonnegative matrix for any positive integer $k \geq 1$, to derive new lower and upper bounds for the spectral radius of the matrix solely expressed by its average $(k+1)$-row sums. For successive values of k, we form sequences with terms these bounds and examine their basic properties of monotonicity and convergence. In Section 3, we generalize the results in [2,9,16,24] to propose a new lower bound for the spectral radius of a nonnegative matrix in terms of its smallest diagonal and off-diagonal elements and its average $(k+1)$-row sums. Based on these quantities, we construct another sequence of lower bounds with respect to k, which sharpen the associated bounds exploited in the previous section and give closer approximations to the spectral radius as k increases. Section 4 is devoted to an analogous methodology on a sequence of upper bounds whose terms admit a representation in the largest diagonal and off-diagonal elements of the matrix and its average $(k+1)$-row sums. Certain conditions are stated under which the optimal approximation to the spectral radius is found among them, which constitutes a sharper formulae compared to the associated upper bound discussed in Section 2. In all the aforesaid bounds, we handle the cases of equality when the matrix is irreducible. Finally, various numerical examples are presented to confirm our theoretical findings and draw comparisons between the proposed bounds and the earlier formulae.

2. Sequences of bounds for the spectral radius

In this section, we generalize the well-known Frobenius' bounds for the spectral radius of a nonnegative matrix to propose new upper and lower bounds, which depend only on the average $(k+1)$-row sums, as defined in (1.1) for a positive integer $k \geq 1$. Moreover, we use these quantities to formulate a sequence of bounds with respect to k and examine its monotonicity and convergence properties.

We begin by stating the next lemma, which collects the two classical Frobenius' bounds for the spectral radius of a nonnegative matrix, [10,12,20].

Lemma 2.1 *(Frobenius' bounds). Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$.*

(i) [20, Theorem 1.1] *Let $r_i(A)$, $i = 1, \ldots, n$ be the i-th row sums of A, then*

$$\min_{1 \leq i \leq n} \{r_i(A)\} \leq \rho(A) \leq \max_{1 \leq i \leq n} \{r_i(A)\}. \tag{2.1}$$

If A is also irreducible, then either equality holds if and only if $r_1(A) = \cdots = r_n(A)$.

(ii) [12, Theorem 8.1.26] *Let $x \in \mathbb{R}^n$ be a vector with positive components x_i, $i = 1, \ldots, n$, then*

$$\min_{1 \leq i \leq n} \left\{ \frac{1}{x_i} \sum_{j=1}^{n} a_{ij} x_j \right\} \leq \rho(A) \leq \max_{1 \leq i \leq n} \left\{ \frac{1}{x_i} \sum_{j=1}^{n} a_{ij} x_j \right\}. \tag{2.2}$$

If A is also irreducible, then either equality holds if and only if x is an eigenvector of A corresponding to $\rho(A)$.

The next proposition can be viewed as a generalization of the result in Lemma 2.1 to the elements of the k-th power of a nonnegative matrix for any positive integer $k \geq 1$.

Proposition 2.2 *(Generalized Frobenius' bounds). Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ and a vector $x \in \mathbb{R}^n$ with positive components x_i, $i = 1, \ldots, n$. Then, for a fixed integer $k \geq 1$*

$$\min_{1 \leq i \leq n} \left\{ \sqrt[k]{\frac{1}{x_i} \sum_{j=1}^{n} a_{ij}^{(k)} x_j} \right\} \leq \rho(A) \leq \max_{1 \leq i \leq n} \left\{ \sqrt[k]{\frac{1}{x_i} \sum_{j=1}^{n} a_{ij}^{(k)} x_j} \right\}. \tag{2.3}$$

Moreover, if A^k is irreducible, then either equality holds if and only if x is an eigenvector of A corresponding to $\rho(A)$.

Proof. Consider the diagonal matrix $Q = \operatorname{diag}(x_1, x_2, ..., x_n)$ with $x_i > 0$ for all $i = 1, \ldots, n$, then $\tilde{A} = Q^{-1} A^k Q = \left[x_i^{-1} a_{ij}^{(k)} x_j \right]_{i,j=1}^{n} \geq 0$. Applying Lemma 2.1 to $\tilde{A}$ for any

$k \geq 1$ and taking into account the similarity invariance property of the spectral radius, whereby $\rho(\tilde{A}) = \rho(Q^{-1}A^kQ) = \rho(A^k)$, we obtain

$$\begin{gathered}
\min_{1\leq i\leq n}\left\{r_i(\tilde{A})\right\} \leq \rho(\tilde{A}) \leq \max_{1\leq i\leq n}\left\{r_i(\tilde{A})\right\} \\
\min_{1\leq i\leq n}\left\{\frac{1}{x_i}\sum_{j=1}^{n} a_{ij}^{(k)}x_j\right\} \leq \rho(A^k) \leq \max_{1\leq i\leq n}\left\{\frac{1}{x_i}\sum_{j=1}^{n} a_{ij}^{(k)}x_j\right\} \qquad (2.4) \\
\min_{1\leq i\leq n}\left\{\sqrt[k]{\frac{1}{x_i}\sum_{j=1}^{n} a_{ij}^{(k)}x_j}\right\} \leq \rho(A) \leq \max_{1\leq i\leq n}\left\{\sqrt[k]{\frac{1}{x_i}\sum_{j=1}^{n} a_{ij}^{(k)}x_j}\right\}.
\end{gathered}$$

Moreover, if $A^k \geq 0$ is irreducible, then $A \geq 0$ is irreducible, as well. By Lemma 2.1, either equality occurs in (2.4) if and only if $x > 0$ is an eigenvector of A^k corresponding to $\rho(A^k)$. Since the positive eigenvectors of A^k and A are unique up to a scalar multiple, x is an eigenvector of A corresponding to $\rho(A)$. $\square$

The next theorem follows naturally from the above generalized Frobenius' bounds and involves the notion of the average $(k+1)$-row sums of a nonnegative matrix, provided that all its row sums are positive. We remind that if $A \geq 0$ is irreducible and there exists a permutation matrix P such that PAP^T is partitioned in the form

$$\begin{bmatrix} 0 & A_{12} & 0 & \cdot & 0 \\ 0 & 0 & A_{23} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & A_{q-1,q} \\ A_{q1} & 0 & \cdot & \cdot & 0 \end{bmatrix}, \qquad (2.5)$$

where all the diagonal blocks are square zero matrices, then it is called *q-cyclic* and the largest positive integer q for which A is q-cyclic is called the *cyclic index* of A. Thus, we are led to an extension of Lemma 2.3 in [2].

Theorem 2.3. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with all row sums positive and $w_i^{(k+1)}(A)$, $i = 1,\ldots,n$, be the i-th average $(k+1)$-row sum of A with $k \geq 1$. Then*

$$\min_{1\leq i\leq n}\left\{\sqrt[k]{w_i^{(k+1)}(A)}\right\} \leq \rho(A) \leq \max_{1\leq i\leq n}\left\{\sqrt[k]{w_i^{(k+1)}(A)}\right\}. \qquad (2.6)$$

If A is also irreducible, then the following hold:

(i) If A^k is irreducible, then either equality holds if and only if the average 2-row sums are equal, i.e. $w_1^{(2)}(A) = \cdots = w_n^{(2)}(A)$.

(ii) If A^k is reducible, then either equality holds if and only if A is k-cyclic and $w_{j_1}^{(2)}(A) = \cdots = w_{j_{n_1}}^{(2)}(A)$, $w_{j_{n_1+1}}^{(2)}(A) = \cdots = w_{j_{n_1+n_2}}^{(2)}(A)$, $\ldots$, $w_{j_{n_1+\cdots+n_k+1}}^{(2)}(A) =$

$\cdots = w^{(2)}_{j_n}(A)$, where the k sets of indices $\{j_1, \ldots, j_{n_1}\}$, $\{j_{n_1+1}, \ldots, j_{n_1+n_2}\}$, ..., $\{j_{n_1+\cdots+n_{k-1}+1}, \ldots, j_n\}$ form a partition of $\{1, \ldots, n\}$ conforming to the block cyclic structure of A.

Proof. Plugging the components $x_i = r_i(A) > 0$, $i = 1, \ldots, n$ into (2.3), the quantity $w^{(k+1)}_i(A)$ arises by the definition in (1.1) and thus, (2.6) is evident.

Moreover, let A be also irreducible. If A^k is irreducible, then either equality occurs in (2.3) if and only if $x = [r_1(A) \quad \ldots \quad r_n(A)]^T > 0$ is a positive eigenvector of A corresponding to $\rho(A)$, or equivalently, if and only if $\rho(A) = \frac{1}{r_i}\sum_{j=1}^n a_{ij}r_j(A) = w^{(2)}_i(A)$ for all $i = 1, \ldots, n$.

On the other hand, Theorem 3.4.5 in [7] asserts that A^k is reducible if and only if there exists a permutation matrix P such that $PA^kP^T = C_1 \oplus \cdots \oplus C_k$, with nonnegative and irreducible block matrices C_j, $j = 1, \ldots, k$. Consequently, $\rho(A) = \sqrt[k]{\rho(C_1)} = \cdots = \sqrt[k]{\rho(C_k)}$ and P conforms to a partition $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ of $\{1, 2, \ldots, n\}$, which describes the k-cyclic structure of PAP^T as in the form (2.5). $\square$

Remark 2.4. It is worth mentioning that:

(i) The first part of Theorem 2.3 can be also derived by [8, Corollary 2.4]. The definition of the i-th row sum of A^{k+1} and Lemma 2.3 in [17] result in the formulation

$$\sum_{j=1}^n a_{ij}r_j(A^k) = r_i(A^{k+1}) = \sum_{j=1}^n a^{(k+1)}_{ij} = \sum_{j=1}^n a^{(k)}_{ij} r_j(A),$$

whereby the i-th average $(k+1)$-row sum of A can be written

$$w^{(k+1)}_i(A) = \frac{1}{r_i(A)}\sum_{j=1}^n a^{(k)}_{ij} r_j(A) = \frac{1}{r_i(A)}\sum_{j=1}^n a^{(k+1)}_{ij} = \frac{r_i(A^{k+1})}{r_i(A)}. \tag{2.7}$$

Now, the compound inequality in (2.6) stems from Corollary 2.4 in [8] setting $k = 1$ and $L = k$ therein and using the expansion of $w^{(k+1)}_i(A)$ in (2.7).

(ii) The second part of Theorem 2.3 stated for irreducible matrices can be also achieved by taking Corollary 3.1 and Theorem 3.3 in [8].

Proposition 2.5. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with all row sums positive and $w^{(k+1)}_i(A)$ be the i-th average $(k+1)$-row sum of A with $k \geq 1$. If A has diagonal elements $a_{ii} \geq 1$, $i = 1, 2, \ldots, n$, then for a fixed index i the sequence $\left\{w^{(k+1)}_i(A)\right\}_{k\geq 1}$ is monotonically increasing. Otherwise, if $0 \leq a_{ii} \leq 1$, $i = 1, 2, \ldots, n$, then $\left\{w^{(k+1)}_i(A)\right\}_{k\geq 1}$ is monotonically decreasing.*

Proof. Assume $A \geq 0$ has diagonal elements $a_{ii} \geq 1$ for all $i = 1, 2, \ldots, n$. Since $r_i(A^{k+1}) \leq a_{ii} r_i(A^{k+1}) + \sum_{j=1, j\neq i}^{n} a_{ij} r_j(A^{k+1}) = \sum_{j=1}^{n} a_{ij} r_j(A^{k+1})$, employing Remark 2.4 for a fixed index i and for any $k \geq 1$ arises

$$w_i^{(k+1)}(A) = \frac{r_i(A^{k+1})}{r_i(A)} \leq \frac{1}{r_i(A)} \sum_{j=1}^{n} a_{ij} r_j(A^{k+1}) = \frac{1}{r_i(A)} \sum_{j=1}^{n} a_{ij}^{(k+1)} r_j(A) = w_i^{(k+2)}(A).$$

Thus, the monotonicity is verified. If $0 \leq a_{ii} \leq 1$, $i = 1, 2, \ldots, n$, the opposite monotonicity of $\left\{ w_i^{(k)}(A) \right\}_{k \geq 1}$ is evident. □

For a given nonnegative matrix and fixed positive components x_i, $i = 1, \ldots, n$, the quantities $\sqrt[k]{\frac{1}{x_i} \sum_{j=1}^{n} a_{ij}^{(k)} x_j}$ depend only on k, so we regard the sequence:

$$\left\{ \sqrt[k]{\frac{1}{x_i} \sum_{j=1}^{n} a_{ij}^{(k)} x_j} \,:\, x_i > 0,\ i = 1, \ldots, n \right\}_{k \geq 1}.$$

The proximity of the terms of the latter sequence to the spectral radius is explored in the following result, giving an additional approximation beyond the generalized Frobenius' bounds in Proposition 2.2.

Theorem 2.6. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ have a positive eigenvector. Then for any vector $x \in \mathbb{R}^n$ with positive components x_i, $i = 1, \ldots, n$,*

$$\rho(A) = \lim_{k \to \infty} \sqrt[k]{\frac{1}{x_i} \sum_{j=1}^{n} a_{ij}^{(k)} x_j}. \tag{2.8}$$

Proof. Consider $\tilde{A} = Q^{-1} A Q$, where $Q = \operatorname{diag}(x_1, x_2, \ldots, x_n) > 0$. If $v \in \mathbb{R}^n$ is a positive eigenvector of A corresponding to $\lambda \in \sigma(A)$, then $\tilde{v} = Q^{-1} v$ is a positive eigenvector of $\tilde{A}$ corresponding to $\lambda \in \sigma(\tilde{A})$. Using [12, Corollary 8.1.33], the row sums of the matrix $\tilde{A}^k = Q^{-1} A^k Q = \left[x_i^{-1} a_{ij}^{(k)} x_j \right]_{i,j=1}^{n} = \left[\tilde{a}_{ij}^{(k)} \right]_{i,j=1}^{n}$ satisfy the inequalities

$$\frac{\min_{1 \leq m \leq n} \{\tilde{v}_m\}}{\max_{1 \leq m \leq n} \{\tilde{v}_m\}} \rho(\tilde{A})^k \leq \sum_{j=1}^{n} \tilde{a}_{ij}^{(k)} \leq \frac{\max_{1 \leq m \leq n} \{\tilde{v}_m\}}{\min_{1 \leq m \leq n} \{\tilde{v}_m\}} \rho(\tilde{A})^k$$

$$\rho(\tilde{A}) \sqrt[k]{\frac{\min_{1 \leq m \leq n} \{\tilde{v}_m\}}{\max_{1 \leq m \leq n} \{\tilde{v}_m\}}} \leq \sqrt[k]{\sum_{j=1}^{n} \tilde{a}_{ij}^{(k)}} \leq \rho(\tilde{A}) \sqrt[k]{\frac{\max_{1 \leq m \leq n} \{\tilde{v}_m\}}{\min_{1 \leq m \leq n} \{\tilde{v}_m\}}}$$

for all $i = 1, \ldots, n$. Both sides of the latter inequality converge to the spectral radius $\rho(\tilde{A}) = \rho(A)$ as $k \to \infty$, since $\tilde{v} > 0$ and $\lim_{k\to\infty} \sqrt[k]{\frac{\min_{1\le m\le n} \{\tilde{v}_m\}}{\max_{1\le m\le n} \{\tilde{v}_m\}}} = \lim_{k\to\infty} \sqrt[k]{\frac{\max_{1\le m\le n} \{\tilde{v}_m\}}{\min_{1\le m\le n} \{\tilde{v}_m\}}} = 1$. Therefore, by the squeeze theorem, we obtain the desired limit in (2.8). □

We present a variant of Theorem 2.6 involving the notion of the average $(k+1)$-row sums of a nonnegative matrix.

Proposition 2.7. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \ge 0$ with all row sums positive and $w_i^{(k+1)}(A)$, $i = 1, \ldots, n$, be the i-th average $(k+1)$-row sum of A with $k \ge 1$. If A has a positive eigenvector, then*

$$\rho(A) = \lim_{k\to\infty} \sqrt[k]{w_i^{(k+1)}(A)}, \quad \text{for any } i = 1, \ldots, n. \tag{2.9}$$

Proof. Setting $x_i = r_i(A) > 0$ for all $i = 1, \ldots, n$ into (2.8), the i-th average $(k+1)$-row sum $w_i^{(k+1)}(A)$ defined by (1.1) arises and the desired limit follows naturally. □

Proposition 2.8. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \ge 0$ with all row sums positive and $w_i^{(k+1)}(A)$, $i = 1, \ldots, n$, be the i-th average $(k+1)$-row sum of A with $k \ge 1$. Then the sequence $\left\{\max_{1\le i\le n}\left\{\sqrt[k]{w_i^{(k+1)}(A)}\right\}\right\}_{k\ge1}$ possesses a monotonically decreasing subsequence*

$$\left\{\max_{1\le i\le n}\left\{\sqrt[b^k]{w_i^{(b^k+1)}(A)}\right\}\right\}_{k\ge1},$$

for any fixed positive integer b.

Proof. The upper bound assertion in (2.3) is related to the maximum row sum matrix norm by

$$\rho(A) \le \sqrt[k]{\max_{1\le i\le n}\left\{\frac{1}{x_i}\sum_{j=1}^{n} a_{ij}^{(k)} x_j\right\}} = \sqrt[k]{|||Q^{-1}A^kQ|||_\infty}, \tag{2.10}$$

where $Q = \operatorname{diag}(x_1, x_2, \ldots, x_n) > 0$. Now, set $x_i = r_i(A) > 0$ in Q and denote $\tilde{A} = Q^{-1}AQ$. Due to (2.10) and the submultiplicative property of the maximum row sum norm, we obtain

$$|||\tilde{A}^{b^{k+1}}|||_\infty = |||\underbrace{\tilde{A}^{b^k}\tilde{A}^{b^k}\cdots\tilde{A}^{b^k}}_{b \text{ times}}|||_\infty \le \left(|||\tilde{A}^{b^k}|||_\infty\right)^b$$

$$\left(|||\tilde{A}^{b^{k+1}}|||_\infty\right)^{\frac{b^{k+1}}{b^{k+1}}} \le \left(|||\tilde{A}^{b^k}|||_\infty\right)^{\frac{b^{k+1}}{b^k}}$$

$$\left(\sqrt[b^{k+1}]{|||\tilde{A}^{b^{k+1}}|||_\infty}\right)^{b^{k+1}} \le \left(\sqrt[b^k]{|||\tilde{A}^{b^k}|||_\infty}\right)^{b^{k+1}}$$
$$\sqrt[b^{k+1}]{|||\tilde{A}^{b^{k+1}}|||_\infty} \le \sqrt[b^k]{|||\tilde{A}^{b^k}|||_\infty}$$

and the assertion is complete. □

3. A lower bound for the spectral radius of nonnegative matrices

In this section, we propose a lower bound for the spectral radius of a nonnegative matrix A involving its lowest quantities s, τ given in (1.3) and its average $(k+1)$-row sums defined in (1.1), which constitutes a sharper bound compared to (2.6) for $k \ge 1$. The proposed lower bound generalizes the associated expressions encountered in [24,16,2] for $k = 1, 2, 3$, respectively.

In the subsequent analysis, we assume the average $(k+1)$-row sums to be arranged so that

$$w_1^{(k+1)}(A) \ge w_2^{(k+1)}(A) \ge \cdots \ge w_n^{(k+1)}(A). \tag{3.1}$$

We commence our study with an auxiliary result that will be used in the sequel to prove our formulae.

Proposition 3.1. *Let $B(x,y) \in \mathcal{M}_n(\mathbb{R})$, $B(x,y) \ge 0$ be defined by*

$$B(x,y) = (x-y)I_n + yJ_n \ne 0, \tag{3.2}$$

where I_n denotes the $n \times n$ identity matrix and J_n denotes an $n \times n$ matrix with all elements equal to 1. Then

$$B^k(x,y) = (\beta_1^{(k)}(x,y) - \beta_2^{(k)}(x,y))I_n + \beta_2^{(k)}(x,y)J_n, \tag{3.3}$$

where

$$\beta_1^{(k)}(x,y) = \frac{1}{n}\left((x+(n-1)y)^k + (n-1)(x-y)^k\right), \tag{3.4}$$
$$\beta_2^{(k)}(x,y) = \frac{1}{n}\left((x+(n-1)y)^k - (x-y)^k\right). \tag{3.5}$$

Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \ge 0$, then

$$0 \le B^k(s,\tau) \le A^k \le B^k(\mu,\nu), \tag{3.6}$$

where μ, ν and s, τ are the extreme elements of A as defined by (1.2) and (1.3), respectively. Either equality holds only if $A = B(\mu,\nu)$ or $A = B(s,\tau)$.

Proof. From (3.2) it is obvious that $B(x,y) \geq 0$, $B(x,y) \neq 0$ is a symmetric matrix with characteristic polynomial

$$\chi_{B(x,y)}(\lambda) = (\lambda - x - (n-1)y)(\lambda - x + y)^{n-1}. \tag{3.7}$$

The eigenspaces of $B(x,y)$ associated to the real eigenvalues $\lambda_1(x,y) = x + (n-1)y$ and $\lambda_2(x,y) = x - y$ are $\mathcal{E}_1 = \operatorname{span}\{\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T\} \subseteq \mathcal{M}_{n,1}(\mathbb{R})$ and $\mathcal{E}_2 = \operatorname{span}\{\begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & -1 & 0 & \dots & 0 \end{bmatrix}^T, \dots, \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & -1 \end{bmatrix}^T\} \subseteq \mathcal{M}_{n,1}(\mathbb{R})$, respectively. For $k \geq 1$ the spectral decomposition of $B(x,y)$ yields

$$B^k(x,y) = VD^k(x,y)V^{-1}, \tag{3.8}$$

where $D(x,y) = \operatorname{diag}(\lambda_1(x,y), \lambda_2(x,y), \dots, \lambda_2(x,y)) \in \mathcal{M}_n(\mathbb{R})$ and $V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \in \mathcal{M}_n(\mathbb{R})$ with $v_1 \in \mathcal{E}_1$ and $v_2, \dots, v_n \in \mathcal{E}_2$.

By (3.8) the elements of $B^k(x,y)$ can be readily computed; the diagonal elements $b_{ii}^{(k)}(x,y)$, $1 \leq i \leq n$, are given by

$$b_{ii}^{(k)}(x,y) = \frac{1}{n}\lambda_1^k(x,y) + \frac{n-1}{n}\lambda_2^k(x,y) = \frac{1}{n}\left((x+(n-1)y)^k + (n-1)(x-y)^k\right)$$

and the off-diagonal elements $b_{ij}^{(k)}(x,y)$, $1 \leq i,j \leq n$, $i \neq j$, are given by

$$b_{ij}^{(k)}(x,y) = \frac{1}{n}\lambda_1^k(x,y) - \frac{1}{n}\lambda_2^k(x,y) = \frac{1}{n}\left((x+(n-1)y)^k - (x-y)^k\right).$$

Notice that the above formulas for the elements of $B^k(x,y)$ yield (3.4) and (3.5) by setting $\beta_1^{(k)}(x,y) \equiv b_{ii}^{(k)}(x,y)$ and $\beta_2^{(k)}(x,y) \equiv b_{ij}^{(k)}(x,y)$, respectively, verifying (3.3).

Now consider any matrix $A \geq 0$, whose largest diagonal and off-diagonal elements are μ, ν and smallest diagonal and off-diagonal elements are s, τ. Plugging the pairs $(x,y) = (\mu,\nu)$ and $(x,y) = (s,\tau)$ into (3.2), the inequality $B(s,\tau) \leq A \leq B(\mu,\nu)$ arises, which implies the inequality for the powers of the matrices [12], i.e., for $k \geq 1$ holds

$$0 \leq B^k(s,\tau) \leq A^k \leq B^k(\mu,\nu),$$

completing the proof of (3.6). Clearly, either equality holds only if $A = B(\mu,\nu)$ or $A = B(s,\tau)$. □

Before we state and prove our main results, we note that for reasons of notational convenience throughout the proofs of our results and whenever there is no confusion of which matrix we refer to, we omit the dependency on A, that is, we set $r_i = r_i(A)$ and $w_i^{(k+1)} = w_i^{(k+1)}(A)$ for $1 \leq i \leq n$.

In the subsequent theorem a new lower bound for the spectral radius is proposed, which depends on the smallest elements s, τ of $A \geq 0$ and on its n-th average row sum $w_n^{(k+1)}(A)$.

Theorem 3.2. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with all row sums positive and average $(k+1)$-row sums $w_i^{(k+1)}(A)$, $i = 1, \ldots, n$ be arranged as in (3.1). Assuming*

$$w_n^{(k+1)}(A) > \delta = \beta_1^{(k)}(s, \tau) - q\beta_2^{(k)}(s, \tau), \tag{3.9}$$

with s, τ the smallest diagonal and off-diagonal elements of A as in (1.3) and q, $\beta_1^{(k)}(s, \tau)$, $\beta_2^{(k)}(s, \tau)$ as defined in (1.4), (3.4), (3.5), respectively, let

$$z_n^{(k+1)} = \frac{1}{2}\left(w_n^{(k+1)}(A) + \delta + \sqrt{\Delta_n}\right), \tag{3.10}$$

where

$$\Delta_n = (w_n^{(k+1)}(A) - \delta)^2 + 4q\beta_2^{(k)}(s, \tau)) \sum_{j=1}^{n-1} (w_j^{(k+1)}(A) - w_n^{(k+1)}(A)). \tag{3.11}$$

Then,

$$\sqrt[k]{z_n^{(k+1)}} \leq \rho(A). \tag{3.12}$$

Proof. If $\tau = 0$, (3.5) results in $\beta_2^{(k)}(s, \tau) = 0$. Thus, (3.10) reduces to

$$z_n^{(k+1)} = \frac{1}{2}\left(w_n^{(k+1)} + \delta + \sqrt{(w_n^{(k+1)} - \delta)^2}\right) = w_n^{(k+1)},$$

due to the assumption $w_n^{(k+1)} > \delta$. Consequently, (3.12) arises straightforwardly by Theorem 2.3. In what follows we assume $\tau > 0$.

Let the $n \times n$ diagonal matrix

$$U = \operatorname{diag}(r_1, \ldots, r_{n-1}, r_n) \operatorname{diag}(x_1, \ldots, x_{n-1}, 1) = RX.$$

The diagonal elements x_j, $j = 1, \ldots, n-1$ of X will be constructed later so that $X \geq I_n$ and in this case we have $U > 0$. Next, we perform the following matrix manipulations on $\tilde{A} = U^{-1}A^kU \geq 0$ after adding and subtracting the nonnegative matrix $R^{-1}A^kR$,

$$\begin{aligned}\tilde{A} &= X^{-1}(R^{-1}A^kRX) = X^{-1}(R^{-1}A^kR + R^{-1}A^kRX - R^{-1}A^kR) \\ &= X^{-1}(R^{-1}A^kR + R^{-1}A^kR(X - I_n)). \end{aligned} \tag{3.13}$$

The left-sided inequality $A^k \geq B^k(s, \tau) = (\beta_1^{(k)}(s, \tau) - \beta_2^{(k)}(s, \tau))I_n + \beta_2^{(k)}(s, \tau)J_n$ in (3.6) applied to (3.13) infers

$$\begin{aligned}\tilde{A} &\geq X^{-1}(R^{-1}A^kR + R^{-1}B^k(s, \tau)R(X - I_n)) \\ &= X^{-1}(R^{-1}A^kR + (\beta_1^{(k)}(s, \tau) - \beta_2^{(k)}(s, \tau))(X - I_n) + \beta_2^{(k)}(s, \tau)R^{-1}J_nR(X - I_n)) \\ &\geq X^{-1}(R^{-1}A^kR + \delta(X - I_n) + q\beta_2^{(k)}(s, \tau)J_n(X - I_n)), \end{aligned} \tag{3.14}$$

since $R^{-1}J_nR \geq (1-q)I_n + qJ_n$, q is given by (1.4) and $\delta = \beta_1^{(k)}(s,\tau) - q\beta_2^{(k)}(s,\tau)$ by (3.9). Noticing that $r_i(R^{-1}A^kR) = w_i^{(k+1)}$, $i = 1, \ldots, n$, inequality (3.14) yields

$$r_i(\tilde{A}) \geq \frac{1}{x_i}\Big(w_i^{(k+1)} + \delta(x_i - 1) + q\beta_2^{(k)}(s,\tau)\sum_{j=1}^{n-1}(x_j - 1)\Big), \quad 1 \leq i \leq n-1 \tag{3.15}$$

$$r_n(\tilde{A}) \geq w_n^{(k+1)} + q\beta_2^{(k)}(s,\tau)\sum_{j=1}^{n-1}(x_j - 1). \tag{3.16}$$

Now, observe that the quadratic equation

$$(z_n^{(k+1)})^2 - (w_n^{(k+1)} + \delta)z_n^{(k+1)} + \delta w_n^{(k+1)} - q\beta_2^{(k)}(s,\tau)\sum_{j=1}^{n-1}(w_j^{(k+1)} - w_n^{(k+1)}) = 0 \tag{3.17}$$

has only real roots, since its discriminant

$$\begin{aligned}\Delta_n &\equiv (w_n^{(k+1)} + \delta)^2 - 4\Big(\delta w_n^{(k+1)} - q\beta_2^{(k)}(s,\tau)\sum_{j=1}^{n-1}(w_j^{(k+1)} - w_n^{(k+1)})\Big)\\ &= (w_n^{(k+1)} - \delta)^2 + 4q\beta_2^{(k)}(s,\tau)\sum_{j=1}^{n-1}(w_j^{(k+1)} - w_n^{(k+1)})\end{aligned}$$

is a positive number, due to $w_n^{(k+1)} > \delta$, $q > 0$, $\beta_2^{(k)}(s,\tau) \geq 0$ and the arrangement of $\left\{w_i^{(k+1)}\right\}_{i=1}^{n}$ in descending order as in (3.1). Hence, a positive real root to (3.17) is

$$z_n^{(k+1)} = \frac{1}{2}\left(w_n^{(k+1)} + \delta + \sqrt{\Delta_n}\right), \tag{3.18}$$

which is used in the determination of

$$x_j = 1 + \frac{w_j^{(k+1)} - w_n^{(k+1)}}{z_n^{(k+1)} - \delta} \Leftrightarrow x_j - 1 = \frac{w_j^{(k+1)} - w_n^{(k+1)}}{z_n^{(k+1)} - \delta}, \quad 1 \leq j \leq n-1. \tag{3.19}$$

Suppose first that $\sum_{j=1}^{n-1}(w_j^{(k+1)} - w_n^{(k+1)}) > 0$, then it is clear by (3.18) that

$$z_n^{(k+1)} > \frac{1}{2}\left(w_n^{(k+1)} + \delta + |w_n^{(k+1)} - \delta|\right) \geq \frac{1}{2}\left(w_n^{(k+1)} + \delta - (w_n^{(k+1)} - \delta)\right) = \delta,$$

otherwise, $w_1^{(k+1)} = \cdots = w_n^{(k+1)} > \delta$ and (3.18) yields

$$z_n^{(k+1)} = \frac{1}{2}\left(w_n^{(k+1)} + \delta + |w_n^{(k+1)} - \delta|\right) > \frac{1}{2}\left(w_n^{(k+1)} + \delta - (w_n^{(k+1)} - \delta)\right) = \delta.$$

Overall, $x_j - 1 \geq 0$ in (3.19) are well defined. Moreover, from (3.17) we derive

$$q\beta_2^{(k)}(s,\tau)\sum_{j=1}^{n-1}(w_j^{(k+1)}-w_n^{(k+1)}) = (z_n^{(k+1)})^2-(w_n^{(k+1)}+\delta)z_n^{(k+1)}+\delta w_n^{(k+1)}$$
$$= (z_n^{(k+1)}-\delta)(z_n^{(k+1)}-w_n^{(k+1)}). \tag{3.20}$$

For $1 \le i,j \le n-1$, we substitute $x_j - 1 \ge 0$ from (3.19) and the expression of $q\beta_2^{(k)}(s,\tau)\sum_{j=1}^{n-1}(w_j^{(k+1)}-w_n^{(k+1)})$ from (3.20) into (3.15) and (3.16) to deduce

$$\begin{aligned}
r_i(\tilde{A}) &\ge \frac{1}{x_i}\Big(w_i^{(k+1)}+q\beta_2^{(k)}(s,\tau)\sum_{j=1}^{n-1}(x_j-1)+\delta(x_i-1)\Big)\\
&= \frac{1}{x_i}\Big(w_i^{(k+1)}+q\beta_2^{(k)}(s,\tau)\sum_{j=1}^{n-1}\frac{w_j^{(k+1)}-w_n^{(k+1)}}{z_n^{(k+1)}-\delta}+\delta\frac{w_i^{(k+1)}-w_n^{(k+1)}}{z_n^{(k+1)}-\delta}\Big)\\
&= \frac{1}{x_i}\Big(w_i^{(k+1)}+\frac{(z_n^{(k+1)}-\delta)(z_n^{(k+1)}-w_n^{(k+1)})}{z_n^{(k+1)}-\delta}+\delta\frac{w_i^{(k+1)}-w_n^{(k+1)}}{z_n^{(k+1)}-\delta}\Big)\\
&= \frac{w_i^{(k+1)}(z_n^{(k+1)}-\delta)+(z_n^{(k+1)}-\delta)(z_n^{(k+1)}-w_n^{(k+1)})+\delta(w_i^{(k+1)}-w_n^{(k+1)})}{x_i(z_n^{(k+1)}-\delta)}\\
&= \frac{(z_n^{(k+1)}-\delta)(z_n^{(k+1)}+w_i^{(k+1)}-w_n^{(k+1)})+\delta(w_i^{(k+1)}-w_n^{(k+1)})}{x_i(z_n^{(k+1)}-\delta)}\\
&= \frac{z_n^{(k+1)}(z_n^{(k+1)}-\delta)+z_n^{(k+1)}(w_i^{(k+1)}-w_n^{(k+1)})}{x_i(z_n^{(k+1)}-\delta)}\\
&= \frac{z_n^{(k+1)}(z_n^{(k+1)}-\delta+w_i^{(k+1)}-w_n^{(k+1)})}{x_i(z_n^{(k+1)}-\delta)}\\
&= \frac{z_n^{(k+1)}(z_n^{(k+1)}-\delta+w_i^{(k+1)}-w_n^{(k+1)})}{\frac{z_n^{(k+1)}-\delta+w_i^{(k+1)}-w_n^{(k+1)}}{z_n^{(k+1)}-\delta}(z_n^{(k+1)}-\delta)} = z_n^{(k+1)},
\end{aligned} \tag{3.21}$$

and (3.16) yields

$$r_n(\tilde{A}) \ge w_n^{(k+1)}+q\beta_2^{(k)}(s,\tau)\sum_{j=1}^{n-1}\frac{w_j^{(k+1)}-w_n^{(k+1)}}{z_n^{(k+1)}-\delta}$$
$$= w_n^{(k+1)}+\frac{(z_n^{(k+1)}-\delta)(z_n^{(k+1)}-w_n^{(k+1)})}{z_n^{(k+1)}-\delta} = z_n^{(k+1)}. \tag{3.22}$$

Hence, both inequalities (3.21) and (3.22) confirm $r_i(\tilde{A}) \ge z_n^{(k+1)}$ for all $1 \le i \le n$. By Lemma 2.1 and the similarity of A^k and $\tilde{A}$, the validity of (3.12) is verified, since

$$\rho(A)^k = \rho(A^k) = \rho(\tilde{A}) \ge \min_{1\le i\le n}\left\{r_i(\tilde{A})\right\} \ge z_n^{(k+1)}. \quad \square \tag{3.23}$$

The following proposition characterizes necessary and sufficient conditions for which the equality part of Theorem 3.2 is attained.

Proposition 3.3. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ be irreducible. Under the notations and assumptions of Theorem 3.2, $\rho(A) = \sqrt[k]{z_n^{(k+1)}}$ if and only if one of the following holds:*

(i) If $\tau = 0$, then $w_1^{(2)}(A) = \cdots = w_n^{(2)}(A)$, when A^k is irreducible. Otherwise, if A^k is reducible, then $w_{j_1}^{(2)}(A) = \cdots = w_{j_{n_1}}^{(2)}(A)$, $w_{j_{n_1+1}}^{(2)}(A) = \cdots = w_{j_{n_1+n_2}}^{(2)}(A)$, $\ldots$, $w_{j_{n_1+\cdots+n_{k-1}+1}}^{(2)}(A) = \cdots = w_{j_n}^{(2)}(A)$, where the k sets of indices $\{j_1, \ldots, j_{n_1}\}$, $\{j_{n_1+1}, \ldots, j_{n_2}\}$, $\ldots$, $\{j_{n_1+\cdots+n_{k-1}+1}, \ldots, j_n\}$ form a partition of $\{1, \ldots, n\}$ describing the block cyclic structure of A.

(ii) If $\tau > 0$, then $w_1^{(k+1)}(A) = \cdots = w_n^{(k+1)}(A)$.

Proof. (i) If $\tau = 0$, then $\rho(A) = \sqrt[k]{z_n^{(k+1)}} = \sqrt[k]{w_n^{(k+1)}(A)}$ and the result is an immediate consequence of Theorem 2.3.

(ii) If $\tau > 0$ suppose first that $\rho(A) = \sqrt[k]{z_n^{(k+1)}}$ with irreducible $A \geq 0$ and consider $\tilde{A} = U^{-1}A^kU \geq 0$ with diagonal matrix $U = RX > 0$, as constructed in the proof of Theorem 3.2. We then distinguish among two cases:

(a) If $A^k \geq 0$ is irreducible, then $\tilde{A}$ is also irreducible. By (3.23), we have

$$z_n^{(k+1)} \leq \min_{1 \leq i \leq n} \left\{ r_i(\tilde{A}) \right\} \leq \rho(\tilde{A}) = \rho(A)^k = z_n^{(k+1)}$$

$$\Rightarrow z_n^{(k+1)} = \min_{1 \leq i \leq n} \left\{ r_i(\tilde{A}) \right\} = \rho(\tilde{A}).$$

By Lemma 2.1(i), $r_1(\tilde{A}) = \cdots = r_n(\tilde{A}) = z_n^{(k+1)}$, then inequalities (3.21), (3.22) and thus, (3.15), (3.16) degenerate to equalities. If $w_1^{(k+1)}(A) > w_n^{(k+1)}(A)$, we consider the smallest integer $2 \leq t \leq n$ such that $w_t^{(k+1)}(A) = w_n^{(k+1)}(A)$. Clearly, $w_i^{(k+1)}(A) > w_n^{(k+1)}(A)$ for integers $1 \leq i \leq t-1$, which imply $x_i > 1$ in (3.19). In this case, the equalities in (3.15) and (3.16) hold only if $A^k = B^k(s,\tau)$ in (3.6) and $R^{-1}J_nR = (1-q)I_n + qJ_n$. Therefore, using the expression (2.7) and the elements of $B^{(k+1)}(s,\tau)$ from (3.3), which are given by $\beta_1^{(k+1)}(s,\tau)$, $\beta_2^{(k+1)}(s,\tau)$ in (3.4), (3.5), we have

$$w_i^{(k+1)}(A) = w_i^{(k+1)}(B(s,\tau)) = \frac{r_i(B^{k+1}(s,\tau))}{r_i(B(s,\tau))} = \frac{(s+(n-1)\tau)^{k+1}}{s+(n-1)\tau} = (s+(n-1)\tau)^k,$$

for all $i = 1, \ldots, n$.

(b) If $A^k \geq 0$ is reducible and so is $\tilde{A}$, there is a permutation matrix P such that

$$PA^kP^T = C_1 \oplus \cdots \oplus C_k$$

with irreducible matrices C_j, $j = 1, \ldots, k$ and $\rho(A) = \sqrt[k]{\rho(C_j)}$, [8]. Clearly,

$$P\tilde{A}P^T = D^{-1}(C_1 \oplus \cdots \oplus C_k)D = B_1 \oplus \cdots \oplus B_k,$$

where $D = PUP^T$ is diagonal and $B_j \geq 0$, $j = 1, \ldots, k$ are $n_j \times n_j$ irreducible matrices with $\rho(B_j) = \rho(C_j)$. By Lemma 2.1

$$z_n^{(k+1)} = \rho(A^k) = \rho(\tilde{A}) = \rho(B_j) \geq \min_{1\leq i\leq n_j} \{r_i(B_j)\} \geq \min_{1\leq i\leq n} \{r_i(\tilde{A})\} = z_n^{(k+1)}$$
$$\Rightarrow \rho(B_j) = \min_{1\leq i\leq n_j} \{r_i(B_j)\}$$

and since $B_j \in \mathcal{M}_{n_j}(\mathbb{R})$ are irrreducible for any $j = 1, \ldots, k$, we have $r_1(B_j) = \cdots = r_{n_j}(B_j) = z_n^{(k+1)}$ for any $j = 1, \ldots, k$. Due to permutational similarity, $r_1(\tilde{A}) = \cdots = r_n(\tilde{A}) = z_n^{(k+1)}$ and (3.15), (3.16) are equalities. Following the same arguments as in (a), we conclude $w_1^{(k+1)}(A) = \cdots = w_n^{(k+1)}(A)$.
Conversely, suppose $w_1^{(k+1)}(A) = \cdots = w_n^{(k+1)}(A)$. Then by (3.10)

$$\sqrt[k]{z_n^{(k+1)}} = \sqrt[k]{w_n^{(k+1)}(A)} = \min_{1\leq i\leq n} \left\{ \sqrt[k]{w_i^{(k+1)}(A)} \right\} = \max_{1\leq i\leq n} \left\{ \sqrt[k]{w_i^{(k+1)}(A)} \right\} = \rho(A),$$

due to Theorem 2.3 and thus, the proof is complete. □

At this point, it is natural to inquire whether the lower bound $\sqrt[k]{z_n^{(k+1)}}$ constructed in Theorem 3.2 is tighter over the average $(k+1)$-row sum $\sqrt[k]{w_n^{(k+1)}(A)}$, which also bounds from below the spectral radius, as argued in Theorem 2.3. This question is addressed in Proposition 3.4, which in fact reveals that $\sqrt[k]{z_n^{(k+1)}}$ improves $\sqrt[k]{w_n^{(k+1)}(A)}$ for each $k \geq 1$.

Proposition 3.4. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$. Suppose the notations and assumptions of Theorem 3.2, then*

$$\sqrt[k]{w_n^{(k+1)}(A)} \leq \sqrt[k]{z_n^{(k+1)}} \leq \rho(A). \tag{3.24}$$

Moreover, if A has a positive eigenvector, then $\rho(A) = \lim_{k\to\infty} \sqrt[k]{z_n^{(k+1)}}$.

Proof. For any positive integer $k \geq 1$ we have

$$4q\beta_2^{(k)}(s,\tau)\sum_{j=1}^{n-1}(w_j^{(k+1)} - w_n^{(k+1)}) \geq 0,$$

due to $q > 0$, $\beta_2^{(k)}(s,\tau) \geq 0$ and the arrangement of $\left\{w_i^{(k+1)}\right\}_{i=1}^{n}$ in descending order as in (3.1). Therefore, the above discussion and the hypothesis $w_n^{(k+1)} > \delta$ deduce

$$z_n^{(k+1)} \geq \frac{1}{2}\left(w_n^{(k+1)} + \delta + \sqrt{(w_n^{(k+1)} - \delta)^2}\right) = w_n^{(k+1)}.$$

The validity of (3.24) is apparent from the latter inequality and (3.12). If we further assume that A has a positive eigenvector, then by Proposition 2.7 and the squeeze theorem we obtain $\rho(A) = \lim_{k\to\infty} \sqrt[k]{z_n^{(k+1)}}$, completing the proof. □

In the subsequent example we verify our findings and we compare the lower bounds for the spectral radius in [1,2,4,9,16,24] with the proposed one in Theorem 3.2 for various values of k. Moreover, for reasons of completeness, we include the classical Frobenius' bounds in Lemma 2.1. The arguments of Proposition 3.4 are confirmed, since Theorem 3.2 gives closer approximations to the spectral radius for any $k \geq 1$ in comparison to Theorem 2.3.

Example 3.5. Consider the matrix $A = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ in [16] with $s = 2$, $\tau = 1$ and spectral radius $\rho(A) = 6$. Ensuring the assumptions (3.1) and (3.9) of Theorem 3.2, for each $k \in \{1, 2, 3, 4, 5, 10, 20, 30\}$ and computing the associated quantities $w_4^{(k+1)}(A)$ and $z_4^{(k+1)}$ from (1.1) and (3.10), we evaluate the lower bounds given by (2.6) and (3.12), respectively. The quantities $\sqrt[k]{z_4^{(k+1)}}$ for the first three values of k coincide with the lower bounds in [24, Theorem 2.3], [16, Theorem 2.3] and [2, Theorem 3.1], in which the average 2-row, average 3-row and average 4-row sums are used, respectively. The remaining values of k refer to the new expression given in (3.12). For reasons of comparison we also present Frobenius' lower bound in (2.1) as well as the ones in [4, Theorem 7], [1, Theorem 10] and [9, Theorem 2.2].

In Table 1 we list all the above values to compare among them and illustrate their performance. As one may observe, Proposition 3.4 is confirmed and $\sqrt[k]{z_4^{(k+1)}}$, $k \geq 5$ reach much closer to the spectral radius rather than the other formulae. Now, in Fig. 1 we draw the first 10 terms (left) and the first 30 terms (right) of $\sqrt[k]{w_4^{(k+1)}(A)}$ and $\sqrt[k]{z_4^{(k+1)}}$ to exhibit their convergence behavior. Since $A > 0$, the assumptions of Propositions 2.7 and 3.4 are guaranteed by Perron's theorem [12, Theorem 8.2.11], and thus we can indeed confirm that both sequences converge to the exact spectral radius represented by a solid line as k increases.

4. Upper bounds for the spectral radius of nonnegative matrices

In the current section, we develop upper bounds for the spectral radius of a nonnegative matrix in terms of its greatest quantities μ, ν given in (1.2) and its average $(k + 1)$-row sums arranged in descending order as in (3.1). In addition, we provide the conditions under which the bound is optimal, in the sense that it is the superior among

Table 1
Numerical comparison of lower bounds for the spectral radius, $\rho(A)$.

Reference	k	Lower Bounds	$\sqrt[k]{w_4^{(k+1)}(A)}$	$\sqrt[k]{z_4^{(k+1)}}$
Frobenius (2.1)		5.0000		
[4, Theorem 7]		5.0000		
[1, Theorem 10]		5.7970		
[9, Theorem 2.2]		5.8284		
[24, Theorem 2.3]	1		5.8000	5.9110
[16, Theorem 2.3]	2		5.8822	5.9380
[2, Theorem 3.1]	3		5.9193	5.9513
Theorem 3.2	4		5.9391	5.9593
Theorem 3.2	5		5.9512	5.9646
Theorem 3.2	10		5.9756	5.9782
Theorem 3.2	20		5.9878	5.9880
Theorem 3.2	30		5.9918	5.9919
$\rho(A) = 6.0000$				

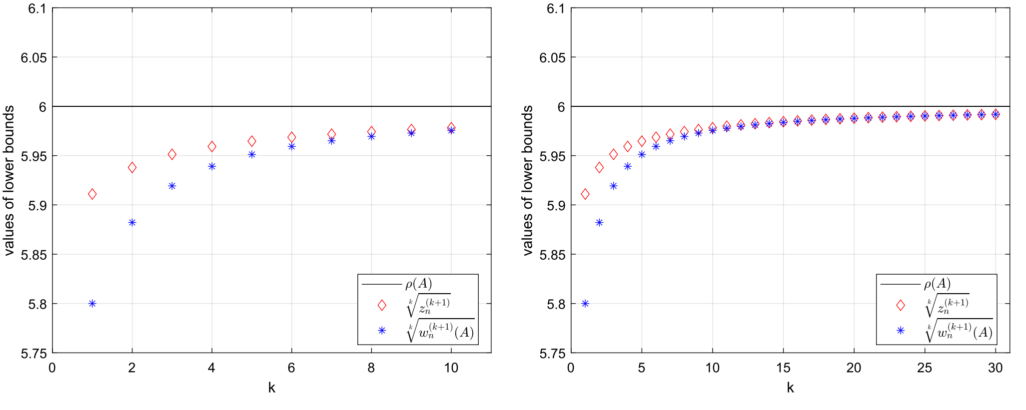

Fig. 1. Lower bounds for $\rho(A)$ consisting of 10 terms (left) and 30 terms (right).

the upper bounds proposed herein and the associated one in Theorem 2.3. The estimates constructed extend the ones encountered in [24,16,2] for $k = 1, 2, 3$, respectively.

As was laid out in Section 3, for brevity of notation throughout the proofs, we omit the dependency on A, that is, we set $r_i = r_i(A)$ and $w_i^{(k+1)} = w_i^{(k+1)}(A)$ for $1 \leq i \leq n$.

Theorem 4.1. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with all row sums positive and average $(k+1)$-row sums $w_i^{(k+1)}(A)$, $i = 1, \dots, n$ be arranged as in (3.1). Denote*

$$\gamma = \beta_1^{(k)}(\mu, \nu) - b\beta_2^{(k)}(\mu, \nu), \tag{4.1}$$

with μ, ν the largest diagonal and off-diagonal elements of A with $\nu > 0$ in (1.2) and b, $\beta_1^{(k)}(\mu, \nu)$, $\beta_2^{(k)}(\mu, \nu)$ defined in (1.4), (3.4), (3.5), respectively. Assuming $w_1^{(k+1)}(A) \geq \gamma$, when $b = 1$, and $w_1^{(k+1)}(A) > \gamma$, when $b > 1$, let

$$Z_\ell^{(k+1)} = \frac{1}{2}\left(w_\ell^{(k+1)}(A) + \gamma + \sqrt{\Delta_\ell}\right), \ \ell = 1, \dots, n, \tag{4.2}$$

where

$$\Delta_\ell = (w_\ell^{(k+1)}(A) - \gamma)^2 + 4b\beta_2^{(k)}(\mu,\nu)\sum_{j=1}^{\ell-1}(w_j^{(k+1)}(A) - w_\ell^{(k+1)}(A)). \tag{4.3}$$

Then,

$$\rho(A) \le \min\left\{ \sqrt[k]{Z_\ell^{(k+1)}} :\ 1 \le \ell \le n \right\}. \tag{4.4}$$

Proof. Consider $\ell = 1$, then the assumption $w_1^{(k+1)} \ge \gamma$ in (4.2) leads to $Z_1^{(k+1)} = w_1^{(k+1)}$ and the result arises straightforwardly by Theorem 2.3.

Consider $2 \le \ell \le n$. If $b = 1$, then $r_1 = \cdots = r_n \Rightarrow w_1^{(k+1)} = \cdots = w_n^{(k+1)}$. Therefore, $Z_\ell^{(k+1)} = w_\ell^{(k+1)} = w_1^{(k+1)}$ for all $\ell = 2, \dots, n$ and Proposition (2.6) implies $\rho(A) = \sqrt[k]{w_1^{(k+1)}} = \sqrt[k]{Z_\ell^{(k+1)}}$. On the other hand, if $b > 1$, let the $n \times n$ diagonal matrix

$$U = \operatorname{diag}(r_1, \dots, r_{\ell-1}, r_\ell, \dots, r_n)\operatorname{diag}(x_1, \dots, x_{\ell-1}, 1, \dots, 1) = RX,$$

in which the diagonal elements $x_j \ge 1$, $j = 1, \dots, \ell-1$ will be determined later and let $\tilde{A} = U^{-1}A^kU \ge 0$. Analogously to the matrix manipulations in (3.13), we have

$$\tilde{A} = X^{-1}(R^{-1}A^kR + R^{-1}A^kR(X - I_n)). \tag{4.5}$$

The right-sided inequality $A^k \le B^k(\mu,\nu) = (\beta_1^{(k)}(\mu,\nu) - \beta_2^{(k)}(\mu,\nu))I_n + \beta_2^{(k)}(\mu,\nu)J_n$ in (3.6) applied to (4.5) infers

$$\begin{aligned}
\tilde{A} &\le X^{-1}(R^{-1}A^kR + R^{-1}B^k(\mu,\nu)R(X - I_n)) \\
&= X^{-1}(R^{-1}A^kR + \beta_2^{(k)}(\mu,\nu)R^{-1}J_nR(X - I_n) + (\beta_1^{(k)}(\mu,\nu) - \beta_2^{(k)}(\mu,\nu))(X - I_n)) \\
&\le X^{-1}(R^{-1}A^kR + b\beta_2^{(k)}(\mu,\nu)J_n(X - I_n) + \gamma(X - I_n)),
\end{aligned} \tag{4.6}$$

since $R^{-1}J_nR \le (1-b)I_n + bJ_n$, and $\gamma = \beta_1^{(k)}(\mu,\nu) - b\beta_2^{(k)}(\mu,\nu)$. Noticing that $r_i(R^{-1}A^kR) = w_i^{(k+1)}$, $i = 1, \dots, n$, and $w_i^{(k+1)} \le w_\ell^{(k+1)}$ for $i = \ell, \dots, n$, inequality (4.6) yields

$$r_i(\tilde{A}) \le \frac{1}{x_i}\Big(w_i^{(k+1)} + b\beta_2^{(k)}(\mu,\nu)\sum_{j=1}^{\ell-1}(x_j - 1) + \gamma(x_i - 1)\Big) \tag{4.7}$$

for $1 \le i \le \ell - 1$ and

$$r_i(\tilde{A}) \le w_i^{(k+1)} + b\beta_2^{(k)}(\mu,\nu)\sum_{j=1}^{\ell-1}(x_j - 1) \le w_\ell^{(k+1)} + b\beta_2^{(k)}(\mu,\nu)\sum_{j=1}^{\ell-1}(x_j - 1) \tag{4.8}$$

for $\ell \leq i \leq n$. Now, we are going to determine the entries x_j of X for $j = 1, 2, \ldots, \ell - 1$ and $\ell = 2, \ldots, n$ by considering the quadratic equations

$$\big(Z_\ell^{(k+1)}\big)^2 - (w_\ell^{(k+1)} + \gamma) Z_\ell^{(k+1)} + \gamma w_\ell^{(k+1)} - b\beta_2^{(k)}(\mu, \nu) \sum_{j=1}^{\ell-1} (w_j^{(k+1)} - w_\ell^{(k+1)}) = 0 \quad (4.9)$$

with discriminant

$$\begin{aligned} \Delta_\ell &\equiv \big(w_\ell^{(k+1)} + \gamma\big)^2 - 4\Big(\gamma w_\ell^{(k+1)} - b\beta_2^{(k)}(\mu, \nu) \sum_{j=1}^{\ell-1} (w_j^{(k+1)} - w_\ell^{(k+1)})\Big) \\ &= (w_\ell^{(k+1)} - \gamma)^2 + 4b\beta_2^{(k)}(\mu, \nu) \sum_{j=1}^{\ell-1} (w_j^{(k+1)} - w_\ell^{(k+1)}). \end{aligned}$$

Due to the hypothesis $w_1^{(k+1)}(A) > \gamma$, when $b > 1$, $\beta_2^{(k)}(\mu, \nu) \geq 0$ and the decreasing arrangement of the nonnegative numbers $\left\{w_i^{(k+1)}\right\}_{i=1}^{n}$, the discriminant Δ_ℓ is positive for all $\ell = 2, \ldots, n$, which yields that the quadratic equations in (4.9) have a positive root

$$Z_\ell^{(k+1)} = \frac{1}{2}\left(w_\ell^{(k+1)} + \gamma + \sqrt{\Delta_\ell}\right), \ \ell = 2, \ldots, n. \quad (4.10)$$

For $1 \leq j \leq \ell - 1$, we set

$$x_j = 1 + \frac{w_j^{(k+1)} - w_\ell^{(k+1)}}{Z_\ell^{(k+1)} - \gamma} \Leftrightarrow x_j - 1 = \frac{w_j^{(k+1)} - w_\ell^{(k+1)}}{Z_\ell^{(k+1)} - \gamma}, \quad (4.11)$$

where $Z_\ell^{(k+1)}$ are given by (4.10). Suppose first that $\sum_{j=1}^{\ell-1}(w_j^{(k+1)} - w_\ell^{(k+1)}) > 0$, then it is clear by relation (4.10) that

$$Z_\ell^{(k+1)} > \frac{1}{2}\Big(w_\ell^{(k+1)} + \gamma + |w_\ell^{(k+1)} - \gamma|\Big) \geq \frac{1}{2}\Big(w_\ell^{(k+1)} + \gamma - (w_\ell^{(k+1)} - \gamma)\Big) = \gamma,$$

otherwise, $w_1^{(k+1)} = \cdots = w_\ell^{(k+1)} > \gamma$ and (4.10) yields

$$Z_\ell^{(k+1)} = \frac{1}{2}\left(w_\ell^{(k+1)} + \gamma + |w_\ell^{(k+1)} - \gamma|\right) = w_\ell^{(k+1)} = w_1^{(k+1)} > \gamma.$$

Both cases ensure $x_j - 1 \geq 0$ and x_j in (4.11) are well defined. Moreover, by (4.9), we may write

$$b\beta_2^{(k)}(\mu, \nu) \sum_{j=1}^{\ell-1} (w_j^{(k+1)} - w_\ell^{(k+1)}) = (Z_\ell^{(k+1)} - \gamma)(Z_\ell^{(k+1)} - w_\ell^{(k+1)}). \quad (4.12)$$

For $1 \le i, j \le \ell - 1$, we substitute $x_j - 1 \ge 0$ from (4.11) and the expression of $b\beta_2^{(k)}(\mu,\nu)\sum_{j=1}^{\ell-1}(w_j^{(k+1)} - w_\ell^{(k+1)})$ from (4.12) into the inequality (4.7) to deduce

$$\begin{aligned}
r_i(\tilde{A}) &\le \frac{1}{x_i}\Big(w_i^{(k+1)} + b\beta_2^{(k)}(\mu,\nu)\sum_{j=1}^{\ell-1}(x_j - 1) + \gamma(x_i - 1)\Big) \\
&= \frac{1}{x_i}\Big(w_i^{(k+1)} + b\beta_2^{(k)}(\mu,\nu)\sum_{j=1}^{\ell-1}\frac{w_j^{(k+1)} - w_\ell^{(k+1)}}{Z_\ell^{(k+1)} - \gamma} + \gamma\frac{w_i - w_\ell}{Z_\ell^{(k+1)} - \gamma}\Big) \\
&= \frac{1}{x_i}\Big(w_i^{(k+1)} + \frac{(Z_\ell^{(k+1)} - \gamma)(Z_\ell^{(k+1)} - w_\ell^{(k+1)})}{Z_\ell^{(k+1)} - \gamma} + \gamma\frac{w_i^{(k+1)} - w_\ell^{(k+1)}}{Z_\ell^{(k+1)} - \gamma}\Big) \\
&= \frac{w_i^{(k+1)}(Z_\ell^{(k+1)} - \gamma) + (Z_\ell^{(k+1)} - \gamma)(Z_\ell^{(k+1)} - w_\ell^{(k+1)}) + \gamma(w_i^{(k+1)} - w_\ell^{(k+1)})}{x_i(Z_\ell^{(k+1)} - \gamma)} \\
&= \frac{(Z_\ell^{(k+1)} - \gamma)Z_\ell^{(k+1)} + (w_i^{(k+1)} - w_\ell^{(k+1)})(Z_\ell^{(k+1)} - \gamma + \gamma)}{x_i(Z_\ell^{(k+1)} - \gamma)} \\
&= \frac{(Z_\ell^{(k+1)} - \gamma)Z_\ell^{(k+1)} + (w_i^{(k+1)} - w_\ell^{(k+1)})Z_\ell^{(k+1)}}{\frac{Z_\ell^{(k+1)} - \gamma + w_i^{(k+1)} - w_\ell^{(k+1)}}{Z_\ell^{(k+1)} - \gamma}(Z_\ell^{(k+1)} - \gamma)} = Z_\ell^{(k+1)}, \ \ell = 2, \dots, n.
\end{aligned} \tag{4.13}$$

Similarly, for $\ell \le i \le n$ and $1 \le j \le \ell - 1$ the inequality (4.8) can be written as

$$\begin{aligned}
r_i(\tilde{A}) &\le w_\ell^{(k+1)} + b\beta_2^{(k)}(\mu,\nu)\sum_{j=1}^{\ell-1}(x_j - 1) = w_\ell^{(k+1)} + \frac{b\beta_2^{(k)}(\mu,\nu)}{Z_\ell^{(k+1)} - \gamma}\sum_{j=1}^{\ell-1}(w_j^{(k+1)} - w_\ell^{(k+1)}) \\
&= w_\ell^{(k+1)} + \frac{(Z_\ell^{(k+1)} - \gamma)(Z_\ell^{(k+1)} - w_\ell^{(k+1)})}{Z_\ell^{(k+1)} - \gamma} = Z_\ell^{(k+1)}.
\end{aligned} \tag{4.14}$$

Thus, for $2 \le \ell \le n$ and $1 \le i \le n$ the inequalities (4.13) and (4.14) verify $r_i(\tilde{A}) \le Z_\ell^{(k+1)}$. By Lemma 2.1 and the similarity of A^k and $\tilde{A}$, we obtain

$$\rho(A)^k = \rho(A^k) = \rho(\tilde{A}) \le \max_{1\le i\le n}\left\{r_i(\tilde{A})\right\} \le Z_\ell^{(k+1)}.$$

Thereby, the desired upper bound in (4.4) is achieved. □

The next proposition establishes necessary and sufficient conditions for equality to occur in (4.4) of Theorem 4.1. We omit the proof, since it is in a similar fashion as the one of Proposition 3.3.

Proposition 4.2. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ be irreducible. Under the notations and assumptions of Theorem 4.1, if we further assume $b > 1$, then $\rho(A) = \sqrt[k]{Z_\ell^{(k+1)}}$, for some $\ell = 1, \dots, n$ if and only if one of the following statements holds:*

(i) For $\ell = 1$, $w_1^{(2)}(A) = \cdots = w_n^{(2)}(A)$, when A^k is irreducible. Otherwise, if A^k is reducible, then $w_{j_1}^{(2)}(A) = \cdots = w_{j_{n_1}}^{(2)}(A)$, $w_{j_{n_1+1}}^{(2)}(A) = \cdots = w_{j_{n_1+n_2}}^{(2)}(A)$, $\dots$, $w_{j_{n_1+\cdots+n_k+1}}^{(2)}(A) = \cdots = w_{j_n}^{(2)}(A)$, where the k sets of indices $\{j_1, \dots, j_{n_1}\}$, $\{j_{n_1+1}, \dots, j_{n_2}\}$, $\dots$, $\{j_{n_1+\cdots+n_{k-1}+1}, \dots, j_n\}$ form a partition of $\{1, \dots, n\}$ describing the block cyclic structure of A.

(ii) For $\ell = 2, \dots, n$, $w_1^{(k+1)}(A) = \cdots = w_n^{(k+1)}(A)$.

Proposition 4.3. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with all row sums positive and average $(k+1)$-row sums $w_i^{(k+1)}(A)$, $i = 1, \dots, n$, be arranged as in (3.1). Suppose the notations and assumptions of Theorem 4.1 and*

$$w_n^{(k+1)}(A) > \gamma = \beta_1^{(k)}(\mu, \nu) - b\beta_2^{(k)}(\mu, \nu). \tag{4.15}$$

Then

$$\sqrt[k]{w_\ell^{(k+1)}(A)} \leq \sqrt[k]{Z_\ell^{(k+1)}}, \ \ell = 1, \dots, n. \tag{4.16}$$

Proof. According to (4.2), we have $Z_1^{(k+1)} = w_1^{(k+1)}$ for any $k \geq 1$. Moreover,

$$4b\beta_2^{(k)}(\mu, \nu) \sum_{j=1}^{\ell-1} \left(w_j^{(k+1)} - w_\ell^{(k+1)} \right) \geq 0,$$

due to $b > 0$, $\beta_2^{(k)}(s, \tau) \geq 0$ and the arrangement of $\left\{ w_j^{(k+1)} \right\}_{j=1}^{n}$ as in (3.1). Now, for $\ell = 2, 3, \dots, n$ the latter inequality and the assumption (4.15) imply

$$Z_\ell^{(k+1)} = \frac{1}{2} \left(w_\ell^{(k+1)} + \gamma + \sqrt{\Delta_\ell} \right) \geq \frac{1}{2} \left(w_\ell^{(k+1)} + \gamma + w_\ell^{(k+1)} - \gamma \right) = w_\ell^{(k+1)}.$$

Consequently, (4.16) arises. □

Now, for a fixed $k \geq 1$ and $\ell = 1, \dots, n$, the positive quantities $\sqrt[k]{Z_\ell^{(k+1)}}$ do not necessarily increase or decrease. In the next theorem we investigate the arrangement pattern of these quantities with respect to ℓ and identify the smallest index at which the minimum value is attained.

Theorem 4.4. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with $A \neq B(\mu, \nu)$, $B(\mu, \nu)$ be defined in (3.2). Suppose the notations and assumptions of Theorem 4.1, then*

$$\min_{1\le\ell\le n} \sqrt[k]{Z_\ell^{(k+1)}} = \sqrt[k]{Z_t^{(k+1)}}, \tag{4.17}$$

where

$$t = \min\Big\{\ell : \sum_{j=1}^{\ell} w_j^{(k+1)}(A) < \ell^2 b\beta_2^{(k)}(\mu,\nu) + \ell\gamma \ \text{ and } \\ w_\ell^{(k+1)}(A) < \ell b\beta_2^{(k)}(\mu,\nu) + \gamma,\ \ 1 \le \ell \le n\Big\}. \tag{4.18}$$

Proof. According to (4.3), we derive

$$\Delta_{\ell+1} - \Delta_\ell = \left(w_\ell^{(k+1)} - w_{\ell+1}^{(k+1)}\right)\left(2\gamma + 4b\ell\beta_2^{(k)}(\mu,\nu) - w_\ell^{(k+1)} - w_{\ell+1}^{(k+1)}\right) \tag{4.19}$$

for $1 \le \ell \le n$. It is clear that if $w_\ell^{(k+1)} = w_{\ell+1}^{(k+1)}$, then $\sqrt[k]{Z_\ell^{(k+1)}} = \sqrt[k]{Z_{\ell+1}^{(k+1)}}$, whereas the converse does not hold. Hence, we assert that $\sqrt[k]{Z_\ell^{(k+1)}} \le \sqrt[k]{Z_{\ell+1}^{(k+1)}}$ if and only if

$$\begin{gathered} w_\ell^{(k+1)} - w_{\ell+1}^{(k+1)} + \sqrt{\Delta_\ell} \le \sqrt{\Delta_{\ell+1}} \quad \Leftrightarrow \\ \left(w_\ell^{(k+1)} - w_{\ell+1}^{(k+1)}\right)^2 + 2\left(w_\ell^{(k+1)} - w_{\ell+1}^{(k+1)}\right)\sqrt{\Delta_\ell} + \Delta_\ell \le \Delta_{\ell+1}. \end{gathered}$$

After some calculations and taking into account the ordering $w_\ell^{(k+1)} > w_{\ell+1}^{(k+1)}$ along with (4.2) and (4.19), we deduce

$$\begin{gathered} w_\ell^{(k+1)} - w_{\ell+1}^{(k+1)} + 2\sqrt{\Delta_\ell} \le 2\gamma + 4b\ell\beta_2^{(k)}(\mu,\nu) - w_\ell^{(k+1)} - w_{\ell+1}^{(k+1)} \quad \Leftrightarrow \\ 0 < \sqrt{\Delta_\ell} \le -w_\ell^{(k+1)} + \gamma + 2b\ell\beta_2^{(k)}(\mu,\nu). \end{gathered} \tag{4.20}$$

By assumption, the right part of (4.20) is a nonnegative quantity, then we can square both sides and conclude

$$\begin{aligned} &(w_\ell^{(k+1)} - \gamma)^2 + 4b\beta_2^{(k)}(\mu,\nu)\sum_{j=1}^{\ell-1}(w_j^{(k+1)} - w_\ell^{(k+1)}) \le \left(2b\ell\beta_2^{(k)}(\mu,\nu) - (w_\ell^{(k+1)} - \gamma)\right)^2 \Leftrightarrow \\ &\sum_{j=1}^{\ell-1} w_j^{(k+1)} - (\ell-1)w_\ell^{(k+1)} \le b\ell^2\beta_2^{(k)}(\mu,\nu) - \ell(w_\ell^{(k+1)} - \gamma) \quad \Leftrightarrow \\ &\sum_{j=1}^{\ell} w_j^{(k+1)} \le \ell^2 b\beta_2^{(k)}(\mu,\nu) + \ell\gamma. \end{aligned}$$

Thus, if $\sum_{j=1}^{\ell} w_j^{(k+1)} \ge \ell\big(\ell b\beta_2^{(k)}(\mu,\nu) + \gamma\big)$, then $\sqrt[k]{Z_\ell^{(k+1)}} \ge \sqrt[k]{Z_{\ell+1}^{(k+1)}}$, and in the case $\sum_{j=1}^{\ell} w_j^{(k+1)} \le \ell\big(\ell b\beta_2^{(k)}(\mu,\nu) + \gamma\big)$, then $\sqrt[k]{Z_\ell^{(k+1)}} \le \sqrt[k]{Z_{\ell+1}^{(k+1)}}$.

Let $1 \le t \le n$ be the smallest integer such that $\sum_{j=1}^{t} w_j^{(k+1)} < t^2 b\beta_2^{(k)}(\mu,\nu) + t\gamma$. For $1 \le \ell \le t-1$, we have by the choice of t that $\sum_{j=1}^{\ell} w_j^{(k+1)} \ge \ell^2 b\beta_2^{(k)}(\mu,\nu) + \ell\gamma$ and thus, $\sqrt[k]{Z_\ell^{(k+1)}} \ge \sqrt[k]{Z_t^{(k+1)}}$. We will prove by induction with respect to ℓ that for $t \le \ell \le n-1$, the strict inequality $\sum_{j=1}^{\ell} w_j^{(k+1)} < \ell^2 b\beta_2^{(k)}(\mu,\nu) + \ell\gamma$ holds. The assertion holds trivially for $\ell = t$ by our choice of t. Suppose that it holds for $t \le \ell \le n-2$. Then $w_{\ell+1}^{(k+1)} \le w_\ell^{(k+1)} < \ell b\beta_2^{(k)}(\mu,\nu) + \gamma$, and hence, by the induction hypothesis,

$$\sum_{j=1}^{\ell+1} w_j^{(k+1)} < \ell\big(\ell b\beta_2^{(k)}(\mu,\nu) + \gamma\big) + \big(\ell b\beta_2^{(k)}(\mu,\nu) + \gamma\big) < (\ell+1)\big((\ell+1) b\beta_2^{(k)}(\mu,\nu) + \gamma\big).$$

Furthermore, the requirement $A \ne B(\mu,\nu)$ with $B(\mu,\nu)$ as defined in (3.2), implies $\sum_{j=1}^{n} w_j^{(k+1)} < n\big(nb\beta_2^{(k)}(\mu,\nu) + \gamma\big)$ and the assertion (4.17) is established. $\square$

Remark 4.5. It is interesting to outline the following observations coming from Theorem 4.4 relatively to the upper bounds suggested in Theorems 2.3 and 4.1.

(i) In the occurrence $t = 1$ in Theorem 4.4, note that the equation (4.18) reduces to the condition $w_1^{(k+1)}(A) < \beta_1^{(k)}(\mu,\nu)$ and the upper bound is

$$\min_{1\le \ell \le n} \left\{ \sqrt[k]{Z_\ell^{(k+1)}} \right\} = \sqrt[k]{Z_1^{(k+1)}} = \sqrt[k]{w_1^{(k+1)}(A)}, \tag{4.21}$$

in which the second equation is explained at the beginning of the proof of Theorem 4.1. In this situation, the largest value of the average $(k+1)$-row sums is superior and there is no point in calculating the other expressions of upper bounds $\sqrt[k]{Z_\ell^{(k+1)}}$, for $\ell = 2, \dots, n$ given by (4.2).

(ii) On the other hand, whenever $t \ne 1$ in Theorem 4.4, we have

$$\min_{1\le \ell \le n} \left\{ \sqrt[k]{Z_\ell^{(k+1)}} \right\} < \sqrt[k]{Z_1^{(k+1)}} = \sqrt[k]{w_1^{(k+1)}(A)}. \tag{4.22}$$

It is notable then that Theorem 4.1 is a refinement for any $k \ge 1$ and the upper bound suggested in (4.4) is superior to the corresponding one stated in the right inequality (2.6).

The forthcoming proposition can be derived directly by combining the relation (4.21) and Proposition 2.7.

Proposition 4.6. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \ge 0$ with $A \ne B(\mu,\nu)$, $B(\mu,\nu)$ be defined in (3.2) and A have a positive eigenvector. Suppose the notations and assumptions of Theorem 4.1 and $w_1^{(k+1)}(A) < \beta_1^{(k)}(\mu,\nu)$, then*

$$\rho(A) = \lim_{k\to\infty} \sqrt[k]{Z_1^{(k+1)}}. \tag{4.23}$$

The numerical example below serves to illustrate the efficiency of our proposed upper bounds comparing among the existing formulae in [1,2,4,9,16,24] and Frobenius' bounds in Lemma 2.1, for the purpose of completeness. In addition, we identify the minimum index t at which $\sqrt[k]{Z_t^{(k+1)}}$ attains the minimum value and consists the tighter upper bound for the spectral radius, as argued in Theorem 4.4. Numerical evidence of the observations in Remark 4.5 is also provided as well as the value of k above which the upper bound for the spectral radius is identical to the maximum average $(k+1)$-row sum in (2.6).

Example 4.7. Consider the positive matrix $B = \begin{bmatrix} 3 & 2 & 3 & 3 \\ 2 & 2 & 2 & 3 \\ 3 & 2 & 3 & 1 \\ 3 & 3 & 3 & 2 \end{bmatrix}$ with maximal elements $\mu = \nu = 3$ and spectral radius $\rho(B) = 10$. To meet the assumptions of Theorem 4.1, we properly arrange $w_\ell^{(k+1)}(B)$, $1 \le \ell \le 4$ as in (3.1) and ensure $w_1^{(k+1)}(B) > \gamma$, with γ in (4.1), for each value of $k \in \{1, 2, 3, 4, 5, 10, 20, 30\}$. Then we compute the upper bounds $w_1^{(k+1)}(B)$ and $\left\{ \sqrt[k]{Z_\ell^{(k+1)}} \right\}_{\ell=1}^{4}$ given by (2.6) and (4.4), respectively. Clearly, $\sqrt[k]{Z_\ell^{(k+1)}}$ reduce to the upper bounds in [24, Theorem 2.1], [16, Theorem 2.1] and [2, Theorem 2.4] for $k = 1, 2, 3$, respectively, while the remaining values of k refer to the proposed upper bounds in Theorem 4.1. For comparisons, we also provide Frobenius' bounds in (2.1), [4, Theorem 9], [1, Theorem 4] and [9, Theorem 2.1].

In order to verify the preceding comment in Remark 4.5, we summarize the performance of all upper bounds for $\rho(B)$ in Table 2. The upper bound $\sqrt[k]{Z_t^{(k+1)}} = \min_{1\le\ell\le4} \left\{ \sqrt[k]{Z_\ell^{(k+1)}} \right\}$ is highlighted in bold and the index t is specified in the last column. Clearly, $\sqrt[k]{Z_1^{(k+1)}} = \sqrt[k]{w_1^{(k+1)}(B)}$. As observed, for $k = 1, \ldots, 7$, the values of $\sqrt[k]{Z_\ell^{(k+1)}}$ appear first to decrease and then to increase with respect to ℓ, which means that $t \ne 1$ and we need to calculate them all so as to determine the lowest value. On the other hand, once $k = 5$ and beyond, $\sqrt[k]{Z_\ell^{(k+1)}}$ increase for $1 \le \ell \le 4$ and thus, $t = 1$.

In pursuit of a visual investigation of the convergence behavior of $\left\{ \sqrt[k]{w_1^{(k+1)}(B)} \right\}_{k=1}^{30}$ and $\left\{ \sqrt[k]{Z_t^{(k+1)}} \right\}_{k=1}^{30}$, we display their graphs at the left part of Fig. 2 by blue "stars" and red "diamonds", respectively. Based on these graphs, we can observe that as k increases both sequences reach very close to the solid line, which represents the spectral radius. Thereby, the limits in (2.9) and (4.23) are indeed confirmed, since, by Perron-Frobenius, $B > 0$ has a positive eigenvector.

Table 2
Numerical comparison of upper bounds for the spectral radius, $\rho(B)$.

Reference	k	Bounds	$\sqrt[k]{w_1^{(k+1)}} \equiv \sqrt[k]{Z_1^{(k+1)}}$	$\sqrt[k]{Z_2^{(k+1)}}$	$\sqrt[k]{Z_3^{(k+1)}}$	$\sqrt[k]{Z_4^{(k+1)}}$	t
Frobenius (2.1)		11.0000					
[4, Theorem 9]		11.0000					
[1, Theorem 4]		10.2143					
[9, Theorem 2.1]		10.1789					
[24, Theorem 2.1]	1		10.1111	10.0978	**10.0405**	10.0411	3
[16, Theorem 2.1]	2		10.0499	10.0445	**10.0290**	10.0310	3
[2, Theorem 2.4]	3		10.0336	10.0304	**10.0280**	10.0306	3
Theorem 4.1	4		10.0252	**10.0232**	10.0283	10.0313	2
Theorem 4.1	5		10.0201	**10.0189**	10.0293	10.0328	2
Theorem 4.1	6		10.0168	**10.0161**	10.0308	10.0346	2
Theorem 4.1	7		10.0144	**10.0141**	10.0325	10.0367	2
Theorem 4.1	8		**10.0126**	10.0127	10.0345	10.0391	1
Theorem 4.1	20		**10.0050**	10.0075	10.0557	10.0635	1
Theorem 4.1	30		**10.0034**	10.0058	10.0519	10.0592	1
$\rho(B) = 10.0000$							

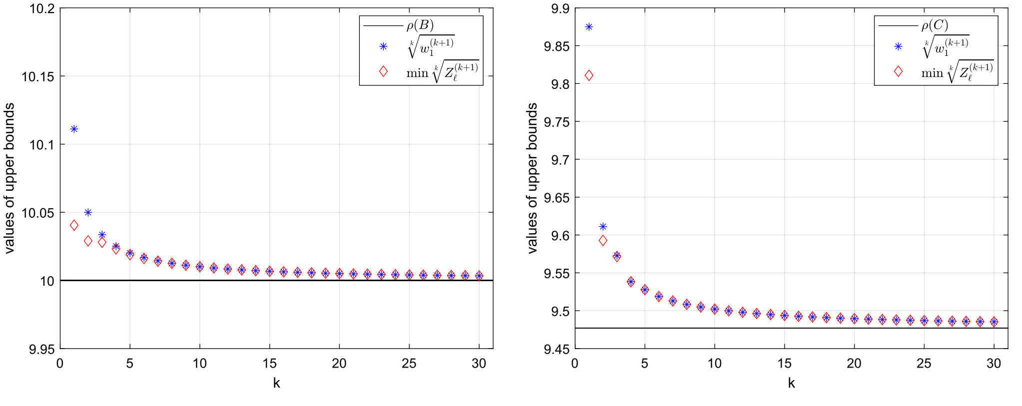

Fig. 2. Upper bounds for $\rho(B)$ (left) and $\rho(C)$ (right) consisting of 30 terms. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Example 4.8. Consider the nonnegative and irreducible matrix $C = \begin{bmatrix} 2 & 1 & 3 & 3 & 2 \\ 3 & 0 & 1 & 1 & 3 \\ 1 & 3 & 0 & 3 & 1 \\ 3 & 3 & 3 & 0 & 3 \\ 1 & 2 & 2 & 3 & 0 \end{bmatrix}$ with $\mu = 2$, $\nu = 3$ and spectral radius $\rho(C) = 9.4769$. Analogously to Example 4.7, we compute all upper bounds and index t and report them in Table 3. Moreover, the right part of Fig. 2 displays the graphs of $\left\{\sqrt[k]{w_1^{(k+1)}(C)}\right\}_{k=1}^{30}$ and $\left\{\sqrt[k]{Z_t^{(k+1)}}\right\}_{k=1}^{30}$ by blue "stars" and red "diamonds", respectively. Notice that the limits in (2.9) and (4.23) also hold here.

Table 3
Numerical comparison of upper bounds for the spectral radius, $\rho(C)$.

Reference	k	Bounds	$\sqrt[k]{Z_1^{(k+1)}}$	$\sqrt[k]{Z_2^{(k+1)}}$	$\sqrt[k]{Z_3^{(k+1)}}$	$\sqrt[k]{Z_4^{(k+1)}}$	$\sqrt[k]{Z_5^{(k+1)}}$	t
Frobenius (2.1)		12.0000						
[4, Theorem 9]		12.0000						
[1, Theorem 4]		10.3263						
[9, Theorem 2.1]		9.9226						
[24, Theorem 2.1]	1		9.8750	9.8750	**9.8108**	9.8119	10.1742	3
[16, Theorem 2.1]	2		9.6112	**9.5928**	9.5936	9.6104	9.8950	2
[2, Theorem 2.4]	3		9.5729	**9.5716**	9.5828	9.6161	9.9460	2
Theorem 4.1	4		9.5384	**9.5382**	9.5414	9.5902	9.9363	2
Theorem 4.1	5		**9.5277**	9.5277	9.5392	9.5925	9.9582	1
Theorem 4.1	10		**9.5021**	9.5026	9.5160	9.5799	9.9667	1
Theorem 4.1	20		**9.4895**	9.4898	9.4982	9.5378	9.7824	1
Theorem 4.1	30		**9.4853**	9.4855	9.4911	9.5176	9.6807	1
$\rho(C) = 9.4769$								

Declaration of competing interest

None declared.

Data availability

No data was used for the research described in the article.

Acknowledgements

This work has received funding by the project "ParICT_CENG: Enhancing ICT research infrastructure in Central Greece to enable processing of Big data from sensor stream, multimedia content, and complex mathematical modeling and simulations" (MIS 5047244), which is implemented under the Action "Reinforcement of the Research and Innovation Infrastructure", funded by the Operational Programme "Competitiveness, Entrepreneurship and Innovation" (NSRF 2014–2020) and co-financed by Greece and the European Union (European Regional Development Fund).

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