Journal of Discrete Mathematical Sciences and Cryptography

ISSN: 0972-0529 (Print) 2169-0065 (Online) Journal homepage: http://www.tandfonline.com/loi/tdmc20

k–step Fibonacci sequences and Fibonacci matrices

Maria Adam & Nicholas Assimakis

To cite this article: Maria Adam & Nicholas Assimakis (2017) k–step Fibonacci sequences and Fibonacci matrices, Journal of Discrete Mathematical Sciences and Cryptography, 20:5, 1183-1206, DOI: 10.1080/09720529.2015.1104929

To link to this article: https://doi.org/10.1080/09720529.2015.1104929

Published online: 22 Nov 2017.

Journal of Discrete Mathematical Sciences & Cryptography
Vol. 20 (2017), No. 5, pp. 1183–1206
DOI : 10.1080/09720529.2015.1104929

k–step Fibonacci sequences and Fibonacci matrices

Maria Adam *
Department of Computer Science and Biomedical Informatics
University of Thessaly
2-4 Papasiopoulou Str.
P.O. 35131
Lamia
Greece

Nicholas Assimakis †
Department of Electronic Engineering
Technological Educational Institute of Central Greece
3rd km Old National Road Lamia-Athens
Lamia
Greece

Abstract

In this paper, the powers of the Fibonacci matrices are investigated and closed formulas for their entries are derived, related to the suitable terms of the k– step Fibonacci sequences in order to develop the properties of the irreducibility and primitivity of the Fibonacci matrices. Some bounds for the spectral radius and the modulus of the remaining eigenvalues of the Fibonacci matrices are presented. New formulas for computing of the odd and even terms of a special Fibonacci sequence are discussed, generalizing the Cassin's and Sharpe's formulas.

Keywords: *Fibonacci numbers and polynomials, Spectral radius, Irreducible, Primitive*

Mathematics Subject Classification: *11B39, 15A18, 15B48*

1. Introduction

Fibonacci numbers are one of the most well-known numbers, and have many important applications to a wide variety of research areas such

*E-mail: `madam@dib.uth.gr`
†E-mail: `assimakis@teiste.gr`

as mathematics, computer science, physics, biology, and statistics. For historical reasons, the applications and the theory of Fibonacci and Lucas numbers, see, e.g. [3, 13, 17, 21, 23] and the references given therein.

It is well-known that the Fibonacci sequence, the Lucas sequence, the Padovan sequence, the Perrin sequence, the tribonacci sequence and the tetranacci sequence are very famous examples of recursive sequences, which are defined as follows.

The Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, . . . are derived by the recurrence relation $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$, with $f_1 = f_2 = 1$, [13], [20, A000045].

The Lucas numbers 2, 1, 3, 4, 7, 11, 18, 29, . . . are derived by the recurrence relation $\ell_n = \ell_{n-1} + \ell_{n-2}$, $n \geq 3$, with $\ell_1 = 2$, and $\ell_2 = 1$, [13], [20, A000032].

The Padovan numbers 1, 0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, . . . are derived by the recurrence relation $a_n = a_{n-2} + a_{n-3}$, $n \geq 4$, with $a_1 = 1$, $a_2 = a_3 = 0$, [20, A000931].

The Perrin numbers 3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, . . . are derived by the recurrence relation $p_n = p_{n-2} + p_{n-3}$, $n \geq 4$, with $p_1 = 3$, $p_2 = 0$, and $p_3 = 2$, [20, A001608].

Both Fibonacci and Lucas numbers as well as both Padovan and Perrin numbers satisfy the same recurrence relation with different initial conditions.

Extending the above definitions, the *k*–step Fibonacci sequences, which have been considered and discussed by many authors [1, 4, 5, 6, 11, 13, 21, 22] and the references therein, are derived. For $k = 3$, the tribonacci numbers 1, 1, 2, 4, 7, 13, 24, 44, . . . are derived by the recurrence relation $f_n = f_{n-1} + f_{n-2} + f_{n-3}$, $n \geq 4$, with $f_1 = f_2 = 1$, and $f_3 = 2$, [4], [22], [20, A000073]. For $k = 4$, the tetranacci numbers 1, 1, 2, 4, 8, 15, 29, 56, . . . are derived by the recurrence relation $f_n = f_{n-1} + f_{n-2} + f_{n-3} + f_{n-4}$, $n \geq 5$, with $f_1 = f_2 = 1$, and $f_3 = 2$, $f_4 = 4$, [20, A000078].

Furthermore, important relations between the *k*–step Fibonacci numbers and the special matrices have been investigated; the determinants of the matrices constructed by *k*–step Fibonacci numbers are obtained in [11], the sums of the generalized Fibonacci numbers are derived directly using the matrix representation and method in [5, 6, 2]; some closed formulas for the associated generalized Fibonacci sequence are derived by matrix methods [12], and recently two limiting properties concerning the *k*-step Fibonacci numbers are obtained, related to the spectral radius of the {0, 1}-matrices in [1]. In the present paper, we shall focus on the important properties of the irreducibility and primitivity of the {0, 1}-matrices through the powers of these matrices, which are related to the k–step Fibonacci numbers.

The paper is organized as follows: In Section 2, we introduce the Fibonacci matrices Q_k, $R_{k,m}$, which correspond to a k–step sum Fibonacci sequence $\left(f_n^{(k,0)}\right)_{n=1,2,\ldots}$, and k–step sum and m–step gap Fibonacci sequence $\left(f_n^{(k,m)}\right)_{n=1,2,\ldots}$, for $k \geq 2$, and $m \geq 1$, respectively. In Section 3, the characteristic polynomials of the Fibonacci matrices are presented and the closed formulas of the entries of the powers of the Fibonacci matrices, which are related to the terms of the Fibonacci sequences, are given. The important properties of the irreducibility and primitivity of the Fibonacci matrices are discussed and bounds for the spectral radius as well as the modulus of the remaining eigenvalues of the Fibonacci matrices are studied. Finally, Section 4 summarizes the conclusions.

2. Definitions of *k*–step Fibonacci sequences and Fibonacci matrices

For the integers $k = 1, 2, \ldots, m = 0, 1, \ldots$, we define the *k–step sum* and *m–step gap Fibonacci sequence* $\left(f_n^{(k,m)}\right)_{n=1,2,\ldots}$ as in [1], whose n–th term is given by the recurrence relation

$$\begin{aligned} f_n &= f_{n-m-1} + f_{n-m-2} + \cdots + f_{n-m-(k-1)} + f_{n-m-k} \\ &= \sum_{i=m+1}^{k+m} f_{n-i}, \text{ for every } n \geq k+m+1, \end{aligned} \tag{1}$$

with

$$f_1 = \ldots = f_{k+m} = 1. \tag{2}$$

Combining (1) and (2) notice that all the terms f_n of the sequence $\left(f_n^{(k,m)}\right)_{n=1,2,\ldots}$ are positive integers and f_n is the sum of k terms starting the sum at the m–th previous term from f_n, thus, the equation (1) can be written equivalently as

$$\begin{aligned} f_n &= f_{n-m-1} + f_{n-m-2} + \cdots + f_{n-m-(k-1)} + f_{n-m-k} \\ &= \sum_{j=1}^{k} f_{n-m-i}, \text{ for every } n \geq k+m+1. \end{aligned} \tag{3}$$

Remark 2.1:

(i) From (2)-(3) it is evident that for $k = 1$, and $m = 0, 1, \ldots$, all the terms of the Fibonacci sequence $\left(f_n^{(k,m)}\right)_{n=1,2,\ldots.}$ are equal to one. Hereafter consider $k \geq 2$, since the case $k = 1$ is trivial.

(ii) For $m = 0$, the n–th term f_n of the k–step sum Fibonacci sequence $\left(f_n^{(k,0)}\right)_{n=1,2,\ldots}$ is given by

$$f_n = f_{n-1} + f_{n-2} + \ldots + f_{n-(k-1)} + f_{n-k} = \sum_{i=1}^{k} f_{n-i}, \text{ for every } n \geq k+1, \tag{4}$$

with initial values

$$f_1 = \ldots = f_k = 1. \tag{5}$$

Remark 2.2: The Fibonacci sequence $\left(f_n^{(k,m)}\right)_{n=1,2,\ldots}$ gives known sequences for various values of the steps *k, m*:

- for $k = 2$, $m = 0$, the equations (2)-(3) give the well-known Fibonacci sequence, 1, 1, 2, 3, 5, 8, 13, . . .
- for $k = 3$, the equations (4)-(5) give the tribonacci sequence, 1, 1, 1, 3, 5, 9, 17, 31, . . ., [20, A000213].
- for $k = 4$, the equations (4)-(5) give the tetranacci sequence, 1, 1, 1, 1, 4, 7, 13, 25, . . ., [20, A000288].
- for $k = 2$, $m = 1$, the equations (2)-(3) give the Padovan sequence, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, . . ., [20, A000931].

In the following, we are going to demonstrate a close link between matrices and Fibonacci numbers in (4) with initial values in (5). To this end, consider $k \geq 2$, and (4) constitute the first equation of the following linear system:

$$\begin{array}{l} f_n = f_{n-1} + f_{n-2} + \ldots + f_{n(k-1)} + f_{n-k} \\ f_{n-1} = f_{n-1} \\ \vdots \\ f_{n-(k-2)} = f_{n-(k-2)} \\ f_{n-(k-1)} = f_{n-(k-1)} \end{array} \Leftrightarrow \begin{bmatrix} f_n \\ f_{n-1} \\ \vdots \\ f_{n-(k-2)} \\ f_{n-(k-1)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_{n-2} \\ \vdots \\ f_{n-(k-1)} \\ f_{n-k} \end{bmatrix}$$

The $k \times k$ matrix Q_k of the coefficients of the above system is defined as

$$Q_k = \begin{bmatrix} \tilde{Q}_1 & 1 \\ I_{k-1} & \tilde{Q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \tag{6}$$

where the $k - 1$ entries of the $1 \times (k - 1)$ vector $\tilde{Q}_1$ are equal to one, I_{k-1} is the $(k - 1) \times (k - 1)$ identity matrix and the $(k - 1) \times 1$ vector $\tilde{Q}_2$ is equal to zero, i.e., Q_k is just the $(k - 1) \times (k - 1)$ identity matrix extended by the first row of k ones and the last column of $k - 1$ zeros. In the following, Q_k is called *k–Fibonacci matrix*.

Working as in the above, for $k \geq 2$, $m \geq 1$ and using (3) with initial values in (2) we can write the following linear system:

$$\begin{aligned} f_n &= f_{n-m-1} + f_{n-m-2} + \cdots + f_{n-m-k+1} + f_{n-m-k} \\ f_{n-1} &= f_{n-1} \\ &\vdots \\ f_{n-m} &= f_{n-m} \\ f_{n-m-1} &= f_{n-m-1} \\ &\vdots \\ f_{n-m-(k-1)} &= f_{n-m-(k-1)} \end{aligned}$$

Hence, using a $(k + m) \times 1$ vector the above linear system can be formed as

$$\begin{bmatrix} f_n \\ f_{n-1} \\ \vdots \\ f_{n-m} \\ f_{n-m-1} \\ \vdots \\ f_{n-m-(k-2)} \\ f_{n-m-(k-1)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & & \vdots \\ \vdots & & & & \ddots & \ddots & & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_{n-2} \\ \vdots \\ f_{n-m} \\ f_{n-m-1} \\ \vdots \\ f_{n-m-(k-1)} \\ f_{n-m-k} \end{bmatrix}$$

whereby it is obvious that the sequence $\left(f_n^{(k,m)}\right)_{n=1,2,\ldots}$ can be represented by a $(k + m) \times (k + m)$ matrix, $R_{k,m}$, which is a block matrix such that

$$R_{k,m} = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & & \vdots \\ \vdots & & & & \ddots & \ddots & & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & & 0 & 1 & 0 \end{bmatrix} \tag{7}$$

where the first row consists of the vector-matrices R_1, R_2; the m entries of the $1 \times m$ vector R_1 are equal to zero and the rest k entries of the $1 \times k$ vector R_2 are equal to one; R_3 is the $(k + m - 1) \times (k + m - 1)$ identity matrix and the $(k + m - 1) \times 1$ vector R_4 is equal to zero. In the following, $R_{k,m}$ is called *k, m – Fibonacci matrix.*

Remark 2.3:

(i) The well-known sequences correspond to k–Fibonacci matrix Q_k in (6) for suitable value of k and $m = 0$;

- for $k = 2$, the Fibonacci sequence corresponds to $Q_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$;

- for $k = 3$, the tribonacci sequence corresponds to $Q_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$;

- for $k = 4$, the tetranacci sequence corresponds to $Q_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(ii) The Padovan sequence corresponds to 2, 1–Fibonacci matrix $R_{2,1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ by (7) with $k = 2$, $m = 1$.

(iii) The k–Fibonacci matrix Q_k in (6) has been presented and the determinant of Q_k has been investigated in [11] and some results on matrices related with Fibonacci numbers and Lucas numbers have been investigated in [7] and the transpose matrix of the general Q–matrix in [10].

(iv) The trace of a matrix A is denoted by $tr(A)$. From (6) and (7), it is evident that $tr(Q_k) = 1$, and $tr(R_{k,m}) = 0$.

3. Fibonacci matrices and powers of Fibonacci matrices

Proposition 3.1: [1] The k–th degree characteristic polynomial $x_{Q_k}(\lambda)$ of the k–Fibonacci matrix Q_k in (6) is given by

$$x_{Q_k}(\lambda) = \lambda^k - \sum_{i=1}^{k} \lambda^{k-i}. \tag{8}$$

The set of all eigenvalues of A is denoted by $\sigma(A)$ and called the *spectrum* of A; the nonnegative real number $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ is called *spectral radius* of A. Recall that a matrix A with nonnegative entries is said to be *primitive*, if A is irreducible and has only one eigenvalue of maximum modulus, [9, Definition 8.5.0].

Here, q_{ij} denotes the ij-th entry of Q_k, for $1 \le i, j \le k$, and $\rho(Q_k)$ the spectral radius of Q_k; in [14, Theorem 3.5] the bounds of $\rho(Q_k)$ have been investigated and presented by

$$\sqrt{\frac{2k-1}{k}} \le \rho(Q_k) < 2. \tag{9}$$

Notice that if $\lambda_j \in \sigma(Q_k)$ is an eigenvalue of Q_k, then $\overline{\lambda}_j \in \sigma(Q_k)$, because of $x_{Q_k}(\lambda)$ has real coefficients. Further, since $x_{Q_k}(\lambda)$ in (8) has the constant term equal to –1, it is evident that

$$\det Q_k = (-1)^k(-1) = (-1)^{k+1}. \tag{10}$$

Hence, Q_k is a nonsingular matrix and all the eigenvalues are nonzero.

In the following proposition, the characteristic polynomial of the k, m–Fibonacci matrix $R_{k,m}$ is formulated.

Proposition 3.2: [1] The $(k + m)$–th degree characteristic polynomial $x_{R_{k,m}}(\lambda)$ of the k, m–Fibonacci matrix $R_{k,m}$ in (7) is given by

$$x_{R_{k,m}}(\lambda) = \lambda^{k+m} - \sum_{i=1}^{k} \lambda^{k-i}. \tag{11}$$

Since $x_{R_{k,m}}(\lambda)$ in (11) has real coefficients, it is evident that if $\lambda_j \in \sigma(R_{k,m})$ is an eigenvalue of $R_{k,m}$, then $\overline{\lambda}_j \in \sigma(R_{k,m})$. Further, using the constant term of $x_{R_{k,m}}(\lambda)$, it is derived

$$\det R_{k,m} = (-1)^{k+m}(-1) = (-1)^{k+m+1}. \tag{12}$$

Hence, $R_{k,m}$ is a nonsingular matrix and all the eigenvalues are nonzero.

Let r_{ij} denote the ij–th entry of $R_{k,m}$, for $1 \le i, j \le k+m$, and $\rho(R_{k,m})$ the spectral radius of $R_{k,m}$; since the values of r_{ij} are 0 or 1, according to [9, Theorem 8.1.22] and [16, Theorem 7] we derive

$$1 = \min_{1\le j\le k+m} \sum_{i=1}^{k+m} r_{ij} < \rho(R_{k,m}) < \max_{1\le j\le k+m} \sum_{i=1}^{k+m} r_{ij} = 2, \tag{13}$$

$$1 = \min_{1\le i\le k+m} \sum_{j=1}^{k+m} r_{ij} < \rho(R_{k,m}) < \max_{1\le i\le k+m} \sum_{j=1}^{k+m} r_{ij} = k, \tag{14}$$

and combining (13) and (14) we can write

$$1 < \rho(R_{k,m}) < 2. \tag{15}$$

In the following, we may rewrite the terms of the Fibonacci sequence $\left(f_n^{(k,m)}\right)_{n=1,2,\ldots}$ such that some initial terms can be defined by negative indexed. To this end, we use the *Dirac delta function* (or δ function), which is denoted by

$$\delta_{n-j} = \begin{cases} 0, & n \ne j \\ 1, & n = j. \end{cases}$$

Consider that, the first $k+m$ negative indexed terms are equal to zero

$$f_{-(k+m-1)} = \cdots = f_{-1} = f_0 = 0, \tag{16}$$

then the n–th number of the Fibonacci sequence $\left(f_n^{(k,m)}\right)_{n=1,2,\ldots}$, which is formulated in the following proposition, follows immediately from (2) and (3).

Proposition 3.3: For the given integers $k \geq 2$, and $m \geq 0$, for all $n \geq 1$ the n–th number, f_n, of the Fibonacci sequence $\left(f_n^{(k,m)}\right)_{n=1,2,\ldots}$ is given by the following recurrence formula

$$f_n = \sum_{i=1}^{k} f_{n-m-i} + \sum_{j=1}^{k+m} \delta_{n-j} - \sum_{j=1}^{k-1} \sum_{i=1}^{k-j} \delta_{n-m-j-i}, \tag{17}$$

with initial values in (16).

By the recurrence formula in (17) and the initial values in (16) for $m = 0$ the n–th number, f_n, of the k–step Fibonacci sequence $\left(f_n^{(k,0)}\right)_{n=1,2,\ldots.}$ is given by

$$f_n = \sum_{i=1}^{k} f_{n-i} + \sum_{j=1}^{k} \delta_{n-j} - \sum_{j=1}^{k-1} \sum_{i=1}^{k-j} \delta_{n-j-i}, \tag{18}$$

with initial values

$$f_{-(k-1)} = \cdots = f_{-1} = f_0 = 0. \tag{19}$$

Consider $\nu \geq 0$ a fixed integer and the k–step Fibonacci sequence $\left(f_n^{(k,0)}\right)_{n=1,2,\ldots.}$. One can easily show, using (4)-(5) or the equivalent expressions (18)-(19), that $(n - \nu + k + 1)$–th term of the k–step Fibonacci sequence is given by

$$f_{n-\nu+k+1} = \sum_{r=1}^{k} f_{n-\nu+r}. \tag{20}$$

Theorem 3.4: *Let the k–Fibonacci matrix Q_k in (6) and the Fibonacci numbers f_n in (18) with initial values in (19). Let $n \geq k$, the n power of Q_k is defined as*

$$Q_k^n = \left[\hat{q}_{ij}\right], \tag{21}$$

where $\hat{q}_{ij}$ denotes the ij–th entry of Q_k^n, for $1 \leq i, j \leq k$. Then, the positive integers $\hat{q}_{ij}$ are given by

$$\hat{q}_{i1} = \frac{1}{k-1} \sum_{r=1}^{k-1} f_{n-i+1+r} \tag{22}$$

$$\hat{q}_{ik} = \frac{1}{k-1} \sum_{r=1}^{k-1} f_{n-i+r} \tag{23}$$

and

$$\hat{q}_{ij} = \frac{1}{k-1}\left(\sum_{r=1}^{j-1} f_{n-i+r} + \sum_{r=1}^{k-1} f_{n-i+1+r}\right), \quad \text{for } 2 \le j \le k-1. \tag{24}$$

Proof: The entries $\hat{q}_{ij}$ are positive integers as sum of the terms the k–step Fibonacci sequence, which is a sequence of positive terms by (18).

Consider $k \ge 2$ a fixed number and use the induction method on n. For $n = k = 2$, and using the Fibonacci numbers by (18) and (19), the entries of matrix in (21) are trivially verified by (22)-(24), since holds

$$Q_2{}^2 = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{12} \\ \hat{q}_{21} & \hat{q}_{22} \end{bmatrix} = \begin{bmatrix} f_3 & f_2 \\ f_2 & f_1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2.$$

Assume that (21) holds for all $n \ge k$, thus the formulas in (22)-(24) are true for n, where n is an arbitrary positive integer less than 2. Consider

$$\begin{aligned} Q_k^{n+1} = Q_k^n Q_k &= \begin{bmatrix} \hat{q}_{11} & \hat{q}_{12} & \hat{q}_{13} & \cdots & \hat{q}_{1k} \\ \hat{q}_{21} & \hat{q}_{22} & \hat{q}_{23} & \cdots & \hat{q}_{2k} \\ \hat{q}_{31} & \hat{q}_{32} & \hat{q}_{33} & \cdots & \hat{q}_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ \hat{q}_{k1} & \hat{q}_{k2} & \hat{q}_{k3} & \cdots & \hat{q}_{kk} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \hat{q}_{11} + \hat{q}_{12} & \hat{q}_{11} + \hat{q}_{13} & \hat{q}_{11} + \hat{q}_{14} & \cdots & \hat{q}_{11} \\ \hat{q}_{21} + \hat{q}_{22} & \hat{q}_{21} + \hat{q}_{23} & \hat{q}_{21} + \hat{q}_{24} & \cdots & \hat{q}_{21} \\ \hat{q}_{31} + \hat{q}_{32} & \hat{q}_{31} + \hat{q}_{33} & \hat{q}_{31} + \hat{q}_{34} & \cdots & \hat{q}_{31} \\ \vdots & \vdots & \vdots & & \vdots \\ \hat{q}_{k1} + \hat{q}_{k2} & \hat{q}_{k1} + \hat{q}_{k1} & \hat{q}_{k1} + \hat{q}_{k4} & \cdots & \hat{q}_{k1} \end{bmatrix} \\ &= \begin{bmatrix} \hat{q}_1 + \hat{q}_2 & \hat{q}_1 + \hat{q}_3 & \hat{q}_1 + \hat{q}_4 & \cdots & \hat{q}_1 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{q}_1 & \tilde{q}_2 & \tilde{q}_3 & \cdots & \tilde{q}_k \end{bmatrix} = \begin{bmatrix} \tilde{q}_{ij} \end{bmatrix} \end{aligned} \tag{25}$$

where $\hat{q}_j, \tilde{q}_j$ denote the j–th column of Q_k^n, Q_k^{n+1}, respectively, and $\tilde{q}_{ij}$ denote the entries of Q_k^{n+1}.

It is obvious that the k–th column of Q_k^{n+1} coincides with the first column of Q_k^n, thus the entries of the k–th column of Q_k^{n+1} are given by (22), i.e.,

$$\tilde{q}_{ik} = \frac{1}{k-1}\sum_{r=1}^{k-1} f_{n-i+1+r} = \frac{1}{k-1}\sum_{r=1}^{k-1} f_{(n+1)-i+r.}$$

Hence, (23) holds also for $n + 1$, which completes the induction method for (23). Since the first column of Q_k^{n+1} is equal to the sum of the first and second column of Q_k^n, the entries of the first column of Q_k^{n+1} are given by (22) and (24) as

$$\tilde{q}_{i1} = \hat{q}_{i1} + \hat{q}_{i2} = \frac{1}{k-1}\sum_{r=1}^{k-1} f_{n-i+1+r} + \frac{1}{k-1}\left(f_{n-i+1} + \sum_{r=2}^{k-1} f_{n-i+1+r} \right)$$

$$= \frac{1}{k-1}\left(\sum_{r=1}^{k-1} f_{(n+1)-i+r} + \sum_{r=2}^{k-1} f_{(n+1)-i+r} + f_{(n+1)-i} \right). \tag{26}$$

In (26) setting $\tau = r - 1$ and using (20) we can write

$$\tilde{q}_{i1} = \frac{1}{k-1}\left(\sum_{\tau=0}^{k-2} f_{(n+2)-i+\tau} + \sum_{\tau=1}^{k-2} f_{(n+2)-i+\tau} + f_{(n+1)-i} \right)$$

$$= \frac{1}{k-1}\left(\sum_{\tau=1}^{k-2} f_{(n+2)-i+\tau} + \sum_{\tau=1}^{k-2} f_{(n+2)-i+\tau} + f_{(n+2)-i} + f_{(n+1)-i} \right)$$

$$= \frac{1}{k-1}\left(\sum_{\tau=1}^{k-2} f_{(n+1)-i+1+\tau} + f_{n-i+1} + f_{n-i+2} + \sum_{\tau=1}^{k-2} f_{n-i+\tau+2} \right)$$

$$= \frac{1}{k-1}\left(\sum_{\tau=1}^{k-2} f_{(n+1)-i+1+\tau} + \sum_{\tau=1}^{k} f_{n-i+\tau} \right)$$

$$= \frac{1}{k-1}\left(\sum_{\tau=1}^{k-2} f_{(n+1)-i+1+\tau} + f_{n-i+k+1} \right) = \frac{1}{k-1}\sum_{\tau=1}^{k-1} f_{(n+1)-i+1+\tau}.$$

Hence, (22) holds also for $n + 1$, which completes the induction method for (22).

By (25) it is clear that, for $2 \le j \le k - 1$ the j–th column of Q_k^{n+1} is equal to the sum of the first and $(j + 1)$–th column of Q_k^n, hence the entries of the j–th column of Q_k^{n+1} are given by (22) and (24) as

$$\tilde{q}_{ij} = \hat{q}_{i1} + \hat{q}_{i,(j+1)} = \frac{1}{k-1}\sum_{r=1}^{k-1} f_{n-i+1+r} + \frac{1}{k-1}\left(\sum_{r=1}^{j} f_{n-i+r} + \sum_{r=j+1}^{k-1} f_{n-i+1+r}\right)$$

$$= \frac{1}{k-1}\left(\sum_{r=1}^{k-1} f_{(n+1)-i+r} + \sum_{r=j+1}^{k-1} f_{(n+1)-i+r} + \sum_{r=1}^{j} f_{n-i+r}\right). \tag{27}$$

Setting $\tau = r - 1$ in (27) and using (20) we can write

$$\tilde{q}_{ij} = \frac{1}{k-1}\left(\sum_{\tau=0}^{k-2} f_{(n+2)-i+\tau} + \sum_{\tau=j}^{k-2} f_{(n+2)-i+\tau} + \sum_{\tau=0}^{j-1} f_{(n+1)-i+\tau}\right)$$

$$= \frac{1}{k-1}\left(\sum_{\tau=0}^{k-2} f_{(n+2)-i+\tau} + f_{(n+2)-i} + \sum_{\tau=j}^{k-2} f_{(n+2)-i+\tau} + \sum_{\tau=1}^{j-1} f_{(n+1)-i+\tau} + f_{(n+1)-i}\right)$$

$$= \frac{1}{k-1}\left(\sum_{\tau=1}^{j-1} f_{(n+1)-i+\tau} + \sum_{\tau=j}^{k-2} f_{(n+1)-i+1+\tau} + f_{n-i+1} + f_{n-i+2} + \sum_{\tau=1}^{k-2} f_{n-i+\tau+2}\right)$$

$$= \frac{1}{k-1}\left(\sum_{\tau=1}^{j-1} f_{(n+1)-i+\tau} + \sum_{\tau=j}^{k-2} f_{(n+1)-i+1+\tau} + \sum_{\tau=1}^{k} f_{n-i+\tau}\right)$$

$$= \frac{1}{k-1}\left(\sum_{\tau=1}^{j-1} f_{(n+1)-i+\tau} + \sum_{\tau=j}^{k-2} f_{(n+1)-i+1+\tau} + f_{n-i+k+1}\right)$$

$$= \frac{1}{k-1}\left(\sum_{\tau=1}^{j-1} f_{(n+1)-i+\tau} + \sum_{\tau=j}^{k-1} f_{(n+1)-i+1+\tau}\right).$$

Hence, (24) holds also for $n + 1$, which completes the induction method for (24). □

The general idea of the Fibonacci cryptography is based on the matrix Q_k^n in (21) for $k = 2$. Applying Theorem 3.4 and using Q_k^n in (21) for various values of $k > 2$, one can provide higher security for encryption and decryption of the initial message, or the image information, [15, 18]. Moreover, Theorem 3.4 will be useful in obtaining two important properties of Q_k, which will lead to the very sharp information for its spectral radius, and a localization of the remain spectrum of Q_k. Indeed, using Theorem 3.4, it is easy to prove that for $k \geq 2$ the entries of the matrix $(I_k + Q_k)^{k-1}$ are positive integers, thus Q_k is an irreducible matrix, [9, Lemma 8.4.1]; it follows that the spectral radius $\rho(Q_k)$ is a positive, simple (without multiplicity) eigenvalue of Q_k, [9, Theorem 8.4.4], which lies in the interval $\left[\left(\frac{2k-1}{k}\right)^{1/2}, 2\right)$ from (9). In addition, applying Theorem 3.4 we conclude that, for $n \geq k$, the entries of Q_k^n are positive integers, thus Q_k is a primitive matrix [9, Theorem 8.5.2], i.e., $\rho(Q_k)$ is the unique eigenvalue with maximum modulus. Thus, if all the distinct eigenvalues of the nonsingular matrix Q_k are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \rho(Q_k)$, then the following inequality holds

$$0 < |\lambda_r| < \rho(Q_k); \quad r = 1, 2, \ldots, k-1. \tag{28}$$

Moreover, applying Lemma 3.4 and Theorem 3.6 (i) in [8] for the characteristic polynomial $x_{Q_k}(\lambda)$ in (8), we conclude that

$$0 < |\lambda_r| < 1; \qquad r = 1, 2, \ldots, k-1. \tag{29}$$

Thus, the following proposition is proved.

Theorem 3.5: *Let Q_k be the k– Fibonacci matrix in (6) with $k \geq 2$. Then, Q_k is an irreducible and primitive matrix, $\rho(Q_k)$ is a positive, simple eigenvalue of Q_k, with $\rho(Q_k) \in (1, 2)$ and the remaining eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ lie in the interior of the unit disk.*

Remark 3.1: Notice that:

(i) If k is odd, then the characteristic polynomial $x_{Q_k}(\lambda)$ in (8) has one real root $\rho(Q_k)$, and the others are complex conjugate. Thus, the complex eigenvalues λ_r in (28) appear in $p = \frac{k-1}{2}$ complex conjugate pairs, which are denoted by $\lambda_1, \lambda_2 = \overline{\lambda}_1, \lambda_3, \lambda_4 = \overline{\lambda}_3, \ldots, \lambda_{p-1}, \lambda_p = \overline{\lambda}_{p-1}$.

(ii) If k is even, then $x_{Q_k}(\lambda)$ has two real roots and the others are complex conjugate. The one real root is the unique real positive root $\rho(Q_k)$, it lies in the interval $\left[\left(\frac{2k-1}{k}\right)^{1/2}, 2\right)$ by (9) and has maximum modulus. Since the degree of $x_{Q_k}(\lambda)$ is even and the constant term of $x_{Q_k}(\lambda)$ is equal to –1, the product all the eigenvalues is equal to –1; thus the other real root is negative, which lies in the interval (–1, 0) due to (29). Hence, the complex eigenvalues λ_r in (28) appear in $p = \frac{k-2}{2}$ complex conjugate pairs and λ_r are denoted as in (i).

(iii) Applying Theorem 3.4 for $k = 2$ and $n \geq k$, the matrix $Q_2^n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{12} \\ \hat{q}_{21} & \hat{q}_{22} \end{bmatrix}$ is given by (22)-(24) as

$$Q_2^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}. \tag{30}$$

Taking the determinant of the both sides of (30), we obtain

$$f_{n+1} f_{n-1} - f_n^2 = (-1)^n, \tag{31}$$

the well-known Cassini's identity [13, Theorem 5.3]. For $k = 3$, the entries of Q_3^n in (21) are given by (22)-(24), thus, Q_3^n relates to the terms of Fibonacci sequence as

$$Q_3^n = \frac{1}{2}\begin{bmatrix} f_{n+1} + f_{n+2} & f_n + f_{n+2} & f_n + f_{n+1} \\ f_n + f_{n+1} & f_{n-1} + f_{n+1} & f_{n-1} + f_n \\ f_{n-1} + f_n & f_{n-2} + f_n & f_{n-2} + f_{n-1} \end{bmatrix}. \tag{32}$$

Since det $Q_3^n = 1$ due to (10), taking the determinant of the both sides of (32), it was found that

$$f_n^3 + f_{n+1}^2 f_{n-2} + f_{n-1}^2 f_{n+2} - f_{n-2} f_n f_{n+2} - 2 f_{n-1} f_n f_{n+1} = 4.$$

In the general case, working as the above and using the formulas in (22)-(24) of Q_k^n and the higher-dimensional determinant, we can derive generalized formula of (31).

(iv) Furthermore for $k = 2$ and $n \geq k$, starting with $Q_2^n = Q_2^{\mu} Q_2^{n-\mu}$, expressing suitably the matrices $Q_2^n, Q_2^{\mu}, Q_2^{n-\mu}$ from (30), we have

$$\begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} = \begin{bmatrix} f_{\mu+1} & f_{\mu} \\ f_{\mu} & f_{\mu-1} \end{bmatrix} \begin{bmatrix} f_{n-\mu+1} & f_{n-\mu} \\ f_{n-\mu} & f_{n-\mu-1} \end{bmatrix},$$

which yields

$$f_n = f_{\mu} f_{n-\mu+1} + f_{\mu-1} f_{n-\mu}, \tag{33}$$

the known as *convolution property* [13, formula 6, p. 88]. In (33) replace n with $2n+1$ and μ with $n+1$, then the well-known Sharpe's identity for $k = 0$ in [19, formula 2] is derived

$$f_{2n+1} = f_{n+1}^2 + f_n^2.$$

Working as the above and using various values of k, we derive the generalized formula of (33).

Example 3.6: Consider $k = 2$, $m = 0$, as in Remark 2.2, and the well-known Fibonacci sequence 1, 1, 2, 3, 5, 8, ... The matrix $Q_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is defined by (6), its characteristic polynomial $x_{Q_2}(\lambda) = \lambda^2 - \lambda - 1$ is given by (8), which has two real roots $\lambda_1 = \frac{1-\sqrt{5}}{2} \approx -0.61803$ and $\rho(Q_2) \equiv \lambda_2 = \frac{1+\sqrt{5}}{2} \approx$ 1.61803, the well-known number as the *golden ratio*.

It is evident that $\rho(Q_2) \in [1.22474,\ 2)$ and $\lambda_1 \in (-1,\ 0)$ verifying the inequalities in (9) and (29) as well as the comments in Remark 3.1 (ii).

Example 3.7: Consider $k = 3$, $m = 0$, as in Remark 2.2, and the well-known tribonacci sequence 1, 1, 1, 3, 5, 9, 17, 31, ... The matrix $Q_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is defined by (6), its characteristic polynomial $x_{Q_3}(\lambda) = \lambda^3 - \lambda^2 - \lambda - 1$ is given by (8), which has roots $\lambda_1 = -0.41964 + 0.60629i$, $\lambda_2 = \bar{\lambda}_1 = -0.41964 - 0.60629i$, and $\rho(Q_3) = 1.83928$. Since $|\lambda_1| = |\lambda_2| = 0.73735$, it is evident that $\rho(Q_3) \in [1.29099,\ 2)$ verifying the inequalities in (9) and (28)-(29). Notice that k is odd, according to Remark 3.1 (i) two eigenvalues are complex conjugate and the spectral radius is the only real one.

Consider $v \geq 0$ a fixed integer and the k–step sum, m–step gap Fibonacci sequence $\left(f_n^{(k,m)}\right)_{n=1,2,\ldots}$, for $k \geq 2$, and $m \geq 1$. One can easily show using the definition (2)-(3) or the equivalent expressions (16)-(17) that, the $(v+m)$–th term of the k–step sum, m–step gap Fibonacci sequence is given by

$$f_{v+m} = \sum_{r=1}^{k} f_{v-r}. \tag{34}$$

Theorem 3.8: *Let $R_{k,m}$ be the k, m–Fibonacci matrix in (7) and f_n be the Fibonacci numbers in (17) with initial values in (16). Let $n \geq k + 2m + 1$, with $k \geq 2$, $m \geq 1$, the n power of $R_{k,m}$ is defined as*

$$R_{k,m}^{n} = \left[\hat{r}_{ij} \right], \tag{35}$$

where $\hat{r}_{ij}$ denotes the ij – th entry of $R_{k,m}^{n}$, for $1 \leq i, j \leq k + m$.

Then, the positive integers $\hat{r}_{ij}$ are given by

$$\hat{r}_{ij} = \frac{1}{k-1}\left(f_{n+k+j-i} - f_{n+j-i} \right), \text{ for } \quad 1 \leq j \leq m, \tag{36}$$

and

$$\hat{r}_{ik} = \frac{1}{k-1} \sum_{r=1}^{k+m+1-j} \left(f_{n+1+k-i-r} - f_{n+1-i-r} \right), \text{ for } \; m+1 \leq j \leq k+m. \tag{37}$$

Proof: It is worth noting that the k–step sum, m–step gap Fibonacci sequence $\left(f_n^{(k,m)} \right)_{n=1,2,\ldots}$ in (17) is an increasing sequence, hence we can write $f_{n+k+j-i} > f_{n+j-i}$ and $f_{n+1+k-i-p} > f_{n+1-i-p}$, for $k \geq 2$, $m \geq 1$, $1 \leq p \leq k + m + 1 - j$, and $1 \leq i, j \leq k + m$. Thus, it is evident that the entries $\hat{r}_{ij}$ in (36) and (37) are positive integers.

Consider $k \geq 2$, and $m \geq 1$ two fixed numbers and use the induction method on n. For $n = k + 2m + 1 = 5$, when $k = 2$, $m = 1$, the Fibonacci numbers by (16) and (17) are 1, 1, 1, 2, 2, 3, 4, 5, 7, . . ., and the entries of matrix in (35) are trivially verified by (36)-(37), since holds

$$R_{2,1}^{5} = \begin{bmatrix} \hat{r}_{11} & \hat{r}_{12} & \hat{r}_{13} \\ \hat{r}_{21} & \hat{r}_{22} & \hat{r}_{23} \\ \hat{r}_{31} & \hat{r}_{32} & \hat{r}_{33} \end{bmatrix} = \begin{bmatrix} f_7 - f_5 & f_6 - f_4 + f_5 - f_3 & f_6 - f_4 \\ f_6 - f_4 & f_5 - f_3 + f_4 - f_2 & f_5 - f_3 \\ f_5 - f_3 & f_4 - f_2 + f_3 - f_1 & f_4 - f_2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{5}.$$

Assume that (35) holds for all $n \geq k + 2m + 1$, thus the formulas in (36)-(37) are true for n, where n is an arbitrary positive integer less than 5. Consider

$$R_{k,m}^{n+1} = R_{k,m}^{n} R_{k,m}$$

$$= \begin{bmatrix} \hat{r}_{11} & \hat{r}_{12} & \hat{r}_{13} & \cdots & \hat{r}_{1,(k+m)} \\ \hat{r}_{21} & \hat{r}_{22} & \hat{r}_{23} & \cdots & \hat{r}_{2,(k+m)} \\ \hat{r}_{31} & \hat{r}_{32} & \hat{r}_{33} & \cdots & \hat{r}_{3,(k+m)} \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \hat{r}_{(k+m),1} & \hat{r}_{(k+m),2} & \hat{r}_{(k+m),3} & \cdots & \hat{r}_{(k+m),(k+m)} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & & & 0 \\ 0 & 1 & 0 & \cdots & & & 0 \\ \vdots & & \ddots & & & & \vdots \\ \vdots & & & & \ddots & & \vdots \\ 0 & \cdots & & \cdots & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{r}_{12} & \hat{r}_{13} & \cdots & \hat{r}_{1m} & \hat{r}_{1,(m+1)} & \hat{r}_{11} + \hat{r}_{1,(m+2)} & \cdots & \hat{r}_{11} + \hat{r}_{1,(k+m)} & \hat{r}_{11} \\ \hat{r}_{22} & \hat{r}_{23} & \cdots & \hat{r}_{2m} & \hat{r}_{2,(m+1)} & \hat{r}_{21} + \hat{r}_{2,(m+2)} & \cdots & \hat{r}_{21} + \hat{r}_{2,(k+m)} & \hat{r}_{21} \\ \hat{r}_{32} & \hat{r}_{33} & \cdots & \hat{r}_{3m} & \hat{r}_{3,(m+1)} & \hat{r}_{31} + \hat{r}_{3,(m+2)} & \cdots & \hat{r}_{31} + \hat{r}_{3,(\mathrm{k+m})} & \hat{r}_{31} \\ \vdots & \vdots & & & & \vdots & & \vdots & \vdots \\ \hat{r}_{(k+m),2} & \hat{r}_{(k+m),3} & \cdots & \hat{r}_{(k+m),m} & \hat{r}_{(k+m),(m+1)} & \hat{r}_{(k+m),1} + \hat{r}_{(k+m),(m+2)} & \cdots & \hat{r}_{(k+m),1} + \hat{r}_{(k+m),(k+m)} & \hat{r}_{(k+m),1} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{r}_2 & \hat{r}_3 & \cdots & \hat{r}_m & \hat{r}_{m+1} & \hat{r}_1 + \hat{r}_{m+2} & \cdots & \hat{r}_1 + \hat{r}_{k+m} & \hat{r}_1 \end{bmatrix} \tag{38}$$

$$= \begin{bmatrix} \tilde{r}_1 & \tilde{r}_2 & \cdots & \tilde{r}_{m-1} & \tilde{r}_m & \tilde{r}_{m+1} & \cdots & \tilde{r}_{k+m-1} & \tilde{r}_{k+m} \end{bmatrix} = \begin{bmatrix} \tilde{r}_{ij} \end{bmatrix}$$

where $\hat{r}_j, \tilde{r}_j$ denote the j–th column of $R_{k,m}^{n}, R_{k,m}^{n+1}$, respectively, and $\tilde{r}_{ij}$ denote the entries of $R_{k,m}^{n+1}$.

By (38) it is obvious that, for $1 \le j \le m-1$, the j–th column of $R_{k,m}^{n+1}$ coincides with the $(j+1)$–th column of $R_{k,m}^{n}$, hence the entries of the j–th column of $R_{k,m}^{n+1}$ are given by (36) as follows

$$\tilde{r}_{ij} = \hat{r}_{i,(j+1)} = \frac{1}{k-1}\left(f_{n+k+(j+1)-i} - f_{n+(j+1)-i}\right) = \frac{1}{k-1}\left(f_{(n+1)+k+j-i} - f_{(n+1)+j-i}\right). \tag{39}$$

Moreover, since the m– th column of $R_{k,m}^{n+1}$ coincides with the $(m+1)$–th column of $R_{k,m}^{n}$ due to (38), using the hypothesis of induction by (37) and (34), we can write

$$\tilde{r}_{im} = \hat{r}_{i,(m+1)} = \frac{1}{k-1}\sum_{r=1}^{k}\left(f_{n+1+k-i-r} - f_{n+1-i-r}\right) = \frac{1}{k-1}\left(f_{(n+1)+k+m-i} - f_{(n+1)+m-i}\right). \tag{40}$$

Hence, for $1 \le j \le m$, (39)-(40) are implied that (36) holds also for $n+1$, which completes the induction method for (36).

By (38) it is clear that, for $m+1 \le j \le k+m-1$, the j–th column of $R_{k,m}^{n+1}$ is equal to the sum of the first and $(j+1)$–th column of $R_{k,m}^{n}$, hence the entries of the j–th column of $R_{k,m}^{n+1}$ are given by (36) and (37) as

$$\tilde{r}_{ij} = \hat{r}_{i1} + \hat{r}_{i,(j+1)} = \frac{1}{k-1}\left(f_{n+k+1-i} - f_{n+1-i}\right) + \frac{1}{k-1}\sum_{r=1}^{k+m+1-(j+1)}\left(f_{n+1+k-i-r} - f_{n+1-i-r}\right)$$

$$= \frac{1}{k-1}\left(f_{n+1+k-i} - f_{n+1-i} + \sum_{r=1}^{k+m-j}\left(f_{n+1+k-i-r} - f_{n+1-i-r}\right)\right)$$

$$= \frac{1}{k-1}\sum_{r=0}^{k+m-j}\left(f_{n+1+k-i-r} - f_{n+1-i-r}\right). \tag{41}$$

Setting $\tau = r+1$ the equation in (41) can be written

$$\tilde{r}_{ij} = \frac{1}{k-1}\sum_{\tau=1}^{k+m-j+1}\left(f_{(n+1)+k-i-(\tau-1)} - f_{(n+1)-i-(\tau-1)}\right)$$

$$= \frac{1}{k-1}\sum_{\tau=1}^{k+m+1-j}\left(f_{(n+1)+1+k-i-\tau} - f_{(n+1)+1-i-\tau}\right). \tag{42}$$

Furthermore, since the $(k+m)$–th column of $R_{k,m}^{n+1}$ coincides with the first column of $R_{k,m}^{n}$ due to (38), the entries of the $(k+m)$–th column of $R_{k,m}^{n+1}$ are given by (36) as

$$\tilde{r}_{i,(k+m)} = \hat{r}_{i1} = \frac{1}{k-1}\left(f_{n+k+1-i} - f_{n+1-i}\right)$$
$$= \frac{1}{k-1}\sum_{r=1}^{k+m+1-(k+m)}\left(f_{(n+1)+1+k-i-r} - f_{(n+1)+1-i-r}\right). \tag{43}$$

Hence, (42)-(43) are implied that (37) holds also for $n+1$, which completes the induction method for (37). □

By (15) the spectral radius $\rho(R_{k,m})$ of $R_{k,m}$ is located in the interval (1, 2) and it is not known if $\rho(R_{k,m})$ is an eigenvalue of $R_{k,m}$ as well as a location for the eigenvalues of $R_{k,m}$. In the following, Theorem 3.8 will be useful in obtaining two important properties of $R_{k,m}$, which will lead to the very sharp information for its spectral radius, that is a real eigenvalue of $R_{k,m}$ and an upper bound for the modulus of the remain spectrum of $R_{k,m}$.

Theorem 3.9: *Let the positive integers k, m, with $k \geq 2$, $m \geq 1$, and let $R_{k,m}$ be the k, m–Fibonacci matrix in (7). Then, $R_{k,m}$ is an irreducible and primitive matrix, $\rho(R_{k,m})$ is a positive, simple eigenvalue of $R_{k,m}$ and the following inequality holds*

$$0 < \left|\lambda_r\right| < \rho(R_{k,m}); \; r = 1, 2, ..., (k + m - 1), \tag{44}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{k+m-1}$ denote the remaining eigenvalues of the nonsingular matrix $R_{k,m}$.

Proof: Using Theorem 3.8, it is easy to prove that for the integers $k \geq 2$, and $m \geq 1$, the entries of the matrix $(I_{k+m} + R_{k,m})^{k+m-1}$ are positive integers, thus $R_{k,m}$ is an irreducible matrix, [9, Lemma 8.4.1]. Moreover, as $R_{k,m}$ has entries 0 or 1, the irreducibility of $R_{k,m}$ follows that the spectral radius $\rho(R_{k,m})$ is a positive, simple eigenvalue of $R_{k,m}$, [9, Theorem 8.4.4].

Furthermore, according to Theorem 3.8 for $n \geq k + 2m + 1$, the entries of $R^n_{k,m}$ are positive integers, thus $R_{k,m}$ is a primitive matrix [9, Theorem 8.5.2], it follows that $\rho(R_{k,m})$ is the unique eigenvalue with maximum modulus. Also, equality (12) implies that $R_{k,m}$ has nonzero eigenvalues i.e.,

$$\lambda_r, \rho(R_{k,m}) \neq 0; \quad r = 1, 2, ..., (k + m - 1). \tag{45}$$

The validity of (44) now follows from the primitivity of $R_{k,m}$ and (45). □

We can now make the following remarks based on the results of Theorem 3.8 and 3.9.

Remark 3.2: Notice that:

(i) If $k+m$ is odd, then the characteristic polynomial $x_{R_{k,m}}(\lambda)$ in (11) has one real root $\rho(R_{k,m})$, and the others are complex conjugate. Thus, the complex eigenvalues λ_r in (45) appear in $p = \dfrac{k+m-1}{2}$ complex conjugate pairs, which are denoted by $\lambda_1, \lambda_2 = \overline{\lambda}_1, \lambda_3, \lambda_4 = \overline{\lambda}_3, \ldots, \lambda_{p-1}$, $\lambda_p = \overline{\lambda}_{p-1}$.

(ii) If $k+m$ is even, then $x_{R_{k,m}}(\lambda)$ has two real roots and the others are complex conjugate. The one real root is the unique real positive root $\rho(R_{k,m})$, it lies in the interval (1, 2) by (15) and has maximum modulus. Since the degree of $x_{R_{k,m}}(\lambda)$ is even and the constant term of $x_{R_{k,m}}(\lambda)$ is equal to –1, the product of all the eigenvalues is equal to –1; thus the other real root is negative, which lies in the real interval $(-\rho(R_{k,m}), 0)$ due to (44). Thus, the complex eigenvalues λ_r in (45) appear in $p = \dfrac{k+m-2}{2}$ complex conjugate pairs and λ_r are denoted as in (i).

(iii) Applying Theorem 3.8, for $k = 2$, $m = 1$, and $n \geq 5$, the entries of the matrix $R_{2,1}^n = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n$ in (35) are given by (36)-(37) as

$$R_{2,1}^n = \begin{bmatrix} \hat{r}_{11} & \hat{r}_{12} & \hat{r}_{13} \\ \hat{r}_{21} & \hat{r}_{22} & \hat{r}_{23} \\ \hat{r}_{31} & \hat{r}_{32} & \hat{r}_{33} \end{bmatrix} = \begin{bmatrix} f_{n+2} - f_n & f_n + f_{n+1} - f_{n-1} - f_{n-2} & f_{n+1} - f_{n-1} \\ f_{n+1} - f_{n-1} & f_{n-1} + f_n - f_{n-2} - f_{n-3} & f_n - f_{n-2} \\ f_n - f_{n-2} & f_{n-2} + f_{n-1} - f_{n-3} - f_{n-4} & f_{n-1} - f_{n-3} \end{bmatrix}.$$

Rewriting the terms f_{n-1}, f_n, f_{n+1}, f_{n+2} of the Fibonacci sequence $\left(f_n^{(2,1)}\right)_{n=1,2,\ldots}$ by (34) for suitable values of v as

$$f_{n-1} = f_{n-4} + f_{n-3},\ f_n = f_{n-2} + f_{n-3},\ f_{n+1} = f_{n-1} + f_{n-2},\ \text{and}\ f_{n+2} = f_{n-1} + f_n,$$

the entries of the above $R_{2,1}^n$ are simplified and $R_{2,1}^n$ is formulated as

$$R_{2,1}^n = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{n-1} & f_n & f_{n-2} \\ f_{n-2} & f_{n-1} & f_{n-3} \\ f_{n-3} & f_{n-2} & f_{n-4} \end{bmatrix}. \tag{46}$$

Taking the determinant of the both sides of (46) and using (12), we obtain

$$f_{n-2}^3 + f_{n-1}^2 f_{n-4} + f_{n-3}^2 f_n - f_{n-4} f_{n-2} f_n - 2 f_{n-3} f_{n-2} f_{n-1} = 1. \tag{47}$$

In the general case, working as the above and using the formulas in (36)-(37) of $R_{k,m}^n$ and the higher-dimensional determinant, we can derive generalized formula of (47).

(iv) Furthermore for $k = 2$, $m = 1$, and $n \geq 5$, starting with $R_{2,1}^n = R_{2,1}^\mu R_{2,1}^{n-\mu}$, expressing suitably the matrices $R_{2,1}^n = R_{2,1}^\mu R_{2,1}^{n-\mu}$ from (46), we have

$$\begin{bmatrix} f_{n-1} & f_n & f_{n-2} \\ f_{n-2} & f_{n-1} & f_{n-3} \\ f_{n-3} & f_{n-2} & f_{n-4} \end{bmatrix} = \begin{bmatrix} f_{\mu-1} & f_\mu & f_{\mu-2} \\ f_{\mu-2} & f_{\mu-1} & f_{\mu-3} \\ f_{\mu-3} & f_{\mu-2} & f_{\mu-4} \end{bmatrix} \begin{bmatrix} f_{n-\mu-1} & f_{n-\mu} & f_{n-\mu-2} \\ f_{n-\mu-2} & f_{n-\mu-1} & f_{n-\mu-3} \\ f_{n-\mu-3} & f_{n-\mu-2} & f_{n-\mu-4} \end{bmatrix},$$

which yields

$$f_n = f_{\mu-1} f_{n-\mu} + f_\mu f_{n-\mu-1} + f_{\mu-2} f_{n-\mu-2}, \tag{48}$$

$$f_{n-2} = f_{\mu-1} f_{n-\mu-2} + f_\mu f_{n-\mu-3} + f_{\mu-2} f_{n-\mu-4}. \tag{49}$$

In (48) replace n with $2n + 1$ and μ with $n + 1$, then the following formula is derived by

$$f_{2n+1} = f_n^2 + f_{n-1} f_{n+1} + f_{n-2} f_{n-1}. \tag{50}$$

In (49) replace n with $2n + 2$ and μ with n, then the following formula is derived by

$$f_{2n} = f_{n-2}^2 + 2 f_{n-1} f_n. \tag{51}$$

Example 3.10: For $k = 2$, $m = 1$, as in Remark 2.2, the corresponded sequence $\left(f_n^{(2,1)}\right)_{n=1,2,\ldots}$ is formed by $f_1 = f_2 = f_3 = 1$, and $f_n = f_{n-3} + f_{n-2}$, for all $n \geq 4$, that is well known as the Padovan sequence 1, 1, 1, 2, 2, 3, 4, 5, . . .

In Remark 2.3, the 3 × 3 matrix $R_{2,1}^n$ is given by

$$R_{2,1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial is given by (11) as $x_{R_{2,1}}(\lambda) = \lambda^3 - \lambda - 1$ and its spectrum

$$\sigma(R_{2,1}) = \left\{ \rho(R_{2,1}) = 1.32472, \lambda_1 = -0.66235 + 0.56227i, \lambda_2 = \overline{\lambda}_1 = -0.66235 - 0.56227i \right\}.$$

Since $|\lambda_1| = |\lambda_2| =$ 0.86883 it is evident that $\rho(R_{2,1}) \in (1, 2)$ and $0 < |\lambda_1| <$ 1.32472 verifying the inequalities in (15) and (44). Notice that $k + m$ is odd, according to Remark 3.2 (i) two eigenvalues are complex conjugate and the spectral radius is the only real eigenvalue.

Example 3.11: Consider the 2–step sum and 2–step gap Fibonacci sequence. Notice that $k + m$ is even. The 4×4 matrix $R_{2,2}$ is given by (7) as

$$R_{2,2} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial is given by (11) as $x_{R_{2,2}}(\lambda) = \lambda^4 - \lambda - 1$ and the eigenvalues of $R_{2,2}$ are $\rho(R_{2,2})$ = 1.22074, λ_1 = –0.24812 + 1.0339i, $\lambda_2 = \overline{\lambda}_1$ = –0.24812 – 1.0339i, and λ_3 = –0.72449. Since $|\lambda_1| = |\lambda_2|$ = 1.06333, $|\lambda_3|$ = 0.72449, it is evident that $\rho(R_{2,2}) \in (1, 2)$ and $0 < |\lambda_{1,2,3}| < \rho(R_{2,2})$ verifying the inequalities in (15) and (44). Notice that $k + m$ is even, according to Remark 3.2 (ii) two eigenvalues are complex conjugate, the spectral radius and λ_3 are the only real eigenvalues; λ_3 is negative and lies in (–1.22074, 0) verifying the comments in Remark 3.2 (ii).

4. Conclusion

In the present paper, the entries of a power of the k–Fibonacci matrix Q_k for the integer $k \geq 2$ corresponded to k–step sum Fibonacci sequence $\left(f_n^{(k,0)}\right)_{n=1,2,\ldots}$, are formulated by the suitable terms of Fibonacci sequence. Moreover, we define the k,m – Fibonacci matrix $R_{k,m}$ for the integers $k \geq 2$, $m \geq 1$ corresponded to k–step sum and m–step gap Fibonacci sequence $\left(f_n^{(k,\mathrm{m})}\right)_{n=1,2,\ldots}$, and the entries of a power of $R_{k,m}$ are formulated by the suitable terms of Fibonacci sequence. We use the formulas of the powers of Q_k, $R_{k,m}$, in order to develop of the important properties of the irreducibility and primitivity of the Fibonacci matrices. Moreover, the spectral radius of the corresponded Fibonacci matrix is a real positive, simple eigenvalue and lies in the interval (1, 2); the remaining eigenvalues of Q_k lie in the interior of the unit disk and the remaining eigenvalues of $R_{k,m}$ lie in the interior of the disk centered at the origin and radius equal to its spectral radius. Finally, using the powers of Q_k, $R_{k,m}$, and special values for k, m, we presented some new formulas for computing f_{2n+1} or f_{2n} numbers related to $f_{n-2}, f_{n-1}, f_n, f_{n+1}$ numbers of the Fibonacci sequences, generalizing the Cassini's or Sharpe's formulas.

References

[1] M. Adam and N. Assimakis, k-step sum and m-step gap Fibonacci sequence, *ISRN Discrete Mathematics*, v. 2014, Article ID 374902, (2014), 7 pages, http://dx.doi.org/10.1155/2014/374902.

[2] M. Adam, N. Assimakis and G. Tziallas, Generalized *k, m*-step Fibonacci sequences and matrices, *Proceedings of the 12th AHA conference,* (2014).

[3] N. Assimakis, M. Adam and C. Triantafillou, Lainiotis filter, golden section and Fibonacci sequence, *Signal Processing,* 93(4), (2013), 721-730.

[4] M. Elia, Derived Sequences, The Tribonacci Recurrence and Cubic Forms, *The Fibonacci Quarterly*, 39(2), (2001), 107-115.

[5] M.C. Er, The matrices of Fibonacci numbers, *The Fibonacci Quarterly*, 22(2), (May, 1984), 134-139.

[6] M.C. Er, Sums of Fibonacci numbers by matrix methods, *The Fibonacci Quarterly*, 22(3), (1984), 204-207.

[7] Xudan Fu and Xia Zhou, On matrices related with Fibonacci and Lucas numbers, *Applied Mathematics and Computation*, 200, (2008), 96-100.

[8] Lixing Han, Michael Neumann and Jianhong Xu, On the roots of certain polynomials arising from the analysis of the NelderMead simplex method, *Linear Algebra and its Applications,* 363, (2003), 109-124.

[9] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 2005.

[10] John Ivie, A general Q-matrix, *The Fibonacci Quarterly*, 10(3), (1972), 255-264.

[11] Erdal Karaduman, An application of Fibonacci numbers in matrices, *Applied Mathematics and Computation*, 147, (2004), 903-908.

[12] Dan Kalman, Generalized Fibonacci numbers by matrix methods, *The Fibonacci Quarterly*, 20(1), (1982), 73-76.

[13] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, Inc., New York, 2001.

[14] G.-Y. Lee, S.-G. Lee and H.-G. Shin, On the k-Generalized Fibonacci matrix Q_k, *Linear Algebra and its Applications*, 251, (1997), 73-88.

[15] M. Mishra, P. Mishra, M.C. Adhikary and S. Kumar, Image encryption using Fibonacci-Lucas transformation, *International Journal on Cryptography and Information Security* (IJCIS), 2(3), (September, 2012), 131-141.

[16] D. Noutsos, Perron-Frobenius theory and some extensions, Como, Italy, May 2008, the presentation available at http://www.math.uoi.gr/ dnoutsos/Papers-pdf-files/slide-perron.pdf.

[17] A.S. Posamentier and I. Lehmann, *The Fabulous Fibonacci Numbers*, Prometheus Books, New York, 2007.

[18] L.-P. Shao, Z. Qin, H.-L.Gao and X.-C. Heng, 2D triangular mappings and their applications in scrambling rectangle image, *Information Technology Journal*, 7(1), (2008), 40-47.

[19] B. Sharpe, On Sums $F_x^2 \pm F_y^2$, *The Fibonacci Quarterly*, 3(1), (Feb. 1965), 63.

[20] N.J.A. Sloane, editor (2003), The On-Line Encyclopedia of Integer Sequences, available at http://oeis.org/.

[21] Predrag Stanimirović, Jovana Nikolov and Ivan Stanimirović, A generalization of Fibonacci and Lucas matrices, *Discrete Applied Mathematics*, 156, (2008), 2606-2619.

[22] Bo Tan and Zhi-Ying Wen, Some properties of the Tribonacci sequence, *European Journal of Combinatorics*, 28, (2007), 1703-1719.

[23] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*, Dover Publications, New York, 2007.

Received July, 2014