Linear Algebra and its Applications 549 (2018) 256–275

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Elliptical higher rank numerical range of some Toeplitz matrices

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A R T I C L E I N F O

Article history:
Received 6 March 2018
Accepted 11 March 2018
Available online 15 March 2018
Submitted by A. Böttcher

MSC:
15A60
15A90
81P68

Keywords:
Rank-k numerical range
Numerical range
Tridiagonal Toeplitz matrix

A B S T R A C T

The higher rank numerical range is described for a class of matrices which happen to be unitarily reducible to direct sums of (at most) 2-by-2 blocks. In particular, conditions are established under which tridiagonal matrices have elliptical rank-k numerical ranges.

1. Introduction

Researchers from the field of theoretical physics implemented several methodologies to resolve problems arising in quantum error correction. The main effort was to eliminate

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https://doi.org/10.1016/j.laa.2018.03.024
0024-3795/

the error factors created during transmission of quantum information and to describe possible corruption induced in the quantum system. Motivated by a physical problem, Choi et al. in their pioneering articles [7–9], reduced this problem to a purely mathematical introducing the notion of higher rank numerical ranges, and triggering the interest of many authors leading to an extensive literature [1,2,16,17,21].

Let $\mathcal{M}_{m,n}(\mathbb{C})$ (resp., $\mathcal{M}_{m,n}(\mathbb{R})$) denote the set of all $m \times n$ complex (resp., real) matrices, with the notation $\mathcal{M}_{n,n}(\mathbb{C})$ abbreviated further to $\mathcal{M}_n(\mathbb{C})$.

For a positive integer $1 \leq k \leq n$, the *rank-k numerical range* of $A \in \mathcal{M}_n(\mathbb{C})$ is defined and denoted by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} \colon PAP = \lambda P \text{ for some rank } k \text{ orthogonal projection } P\}.$$

Note that the rank-1 numerical range coincides with the classical *numerical range* [15]

$$\Lambda_1(A) \equiv F(A) = \{x^*Ax : x \in \mathbb{C}^n,\, x^*x = 1\}\,.$$

The latter set encompasses all the eigenvalues of matrix A, that is the *spectrum* $\sigma(A) = \{\lambda \in \mathbb{C} : \det(\lambda I - A) = 0\}$.

The higher rank numerical ranges $\{\Lambda_k(A)\}_{k\geq 1}$ form a decreasing sequence of compact sets, due to the inclusion relations

$$\Lambda_1(A) \supseteq \Lambda_2(A) \supseteq \cdots \supseteq \Lambda_k(A)$$

and they further enjoy a number of basic algebraic and geometric properties [7,8,16]:

- **($\mathbf{P_1}$)** $\Lambda_k(aA + bI) = a\Lambda_k(A) + b$, for any $a, b \in \mathbb{C}$.
- **($\mathbf{P_2}$)** $\Lambda_k(U^*AU) = \Lambda_k(A)$, for any unitary $U \in \mathcal{M}_n(\mathbb{C})$.
- **($\mathbf{P_3}$)** $\Lambda_k(A) \subseteq \Lambda_k(H(A)) + \mathrm{i}\Lambda_k(S(A))$, where $H(A) = (A + A^*)/2$ and $S(A) = (A - A^*)/2\mathrm{i}$ are the Hermitian and skew-Hermitian parts of matrix A, respectively.
- **($\mathbf{P_4}$)** $\Lambda_k(A_1 \oplus A_2) \supseteq \Lambda_k(A_1) \cup \Lambda_k(A_2)$, where the symbol $\oplus$ stands for the direct sum of matrices $A_1, A_2 \in \mathcal{M}_n(\mathbb{C})$.
- **($\mathbf{P_5}$)** $\Lambda_{k_1+k_2}(A_1 \oplus A_2) \supseteq \Lambda_{k_1}(A_1) \cap \Lambda_{k_2}(A_2)$, for any $k_1, k_2 \in \{1, \ldots, n\}$.
- **($\mathbf{P_6}$)** If $n \geq 3k - 2$, then $\Lambda_k(A) \neq \emptyset$. On the other hand, $\Lambda_n(A) \neq \emptyset$ precisely when $A = \lambda I_n$.

For any $1 \leq k \leq n$, $\Lambda_k(A)$ are convex sets (see [21]). Specifically, they coincide with the intersection of half-planes

$$\Lambda_k(A) = \bigcap_{\theta\in[0,2\pi)} e^{-\mathrm{i}\theta}\{z \in \mathbb{C} : \mathrm{Re}z \leq \lambda_k(H_\theta(A))\}, \tag{1.1}$$

where $\lambda_k(\cdot)$ denotes the k-th largest eigenvalue of a matrix and $H_\theta(A) = H(e^{\mathrm{i}\theta}A)$ (see [17]). In case of a normal matrix A with spectrum $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$, the expression (1.1) yields the intersection of the convex hulls (polygons)

$$\Lambda_k(A) = \bigcap_{1\le j_1<\cdots<j_{n-k+1}\le n} \operatorname{conv}\left\{\lambda_{j_1},\ldots,\lambda_{j_{n-k+1}}\right\}. \tag{1.2}$$

If A is hermitian with non-increasingly ordered (real) eigenvalues, then (1.2) further reduces to the line segment $[\lambda_{n-k+1}, \lambda_k]$.

The goal of this article is to explicitly determine the rank-k numerical range of tridiagonal Toeplitz matrices with special structure appearing in applications. In Section 2, we investigate the rank-k numerical range of the direct sum of 2×2 matrices with fixed diagonal elements. The result obtained is used to extend an elliptical numerical range statement from [4] to an elliptical rank-k numerical range theorem for tridiagonal Toeplitz matrices with periodic entries along their diagonals. Finally, Section 3 addresses the rank-k numerical range of tridiagonal and s-tridiagonal Toeplitz matrices.

2. Direct sum of special 2 × 2 matrices

We start this section studying the rank-k numerical range of the direct sum of 2×2 matrices with fixed diagonal elements. This result will help us generalize the elliptical numerical range theorem for some special matrices elaborated in [4] to the elliptical rank-k numerical range.

Theorem 1. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be unitarily equivalent to the direct sum*

$$cI_{n-2r} \oplus A_1 \oplus \cdots \oplus A_r, \tag{2.1}$$

where $c \in \{a_1, a_2\} \subseteq \mathbb{C}$ and $A_j = \begin{bmatrix} a_1 & s_j \\ t_j & a_2 \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})$, $j = 1,\ldots,r$, are not scalar matrices. If $r < \frac{n}{2}$, then we obtain the following cases:

$$\Lambda_k(A) = \begin{cases} \bigcap_{1\le j_1<\cdots<j_{r-k+1}\le r} \operatorname{conv}\left(F(A_{j_1})\cup\cdots\cup F(A_{j_{r-k+1}})\right), & k\le r \\ \{c\}, & r<k\le n-r \\ \emptyset, & \textit{otherwise.} \end{cases}$$

If $r = \frac{n}{2}$, then

$$\Lambda_k(A) = \begin{cases} \bigcap_{1\le j_1<\cdots<j_{r-k+1}\le r} \operatorname{conv}\left(F(A_{j_1})\cup\cdots\cup F(A_{j_{r-k+1}})\right), & k\le r \\ \emptyset, & r<k\le n. \end{cases}$$

Proof. Due to the unitary invariance property (P2) of the rank-k numerical range, we may assume that A already is in the form (2.1).

Let $r < n/2$, then we will determine the rank-k numerical range by using the equality (1.1) and by computing the k-th largest eigenvalue $\lambda_k(H_\theta(A))$ of the matrix $H_\theta(A) = \operatorname{Re}(e^{i\theta}c)I_{n-2r} \bigoplus_{j=1}^{r} H_\theta(A_j)$ for all values of integer k. It is readily verified that

$$\sigma(H_\theta(A)) = \left\{\mathrm{Re}(e^{\mathrm{i}\theta}c)\right\} \bigcup_{j=1}^{r} \left\{\lambda_1(H_\theta(A_j)), \lambda_2(H_\theta(A_j))\right\},$$

with

$$\lambda_1(H_\theta(A_j)) = \frac{1}{2}\mathrm{Re}(e^{\mathrm{i}\theta}(a_1+a_2)) + \frac{1}{2}\sqrt{(\mathrm{Re}(e^{\mathrm{i}\theta}(a_1-a_2)))^2 + |e^{\mathrm{i}\theta}s_j + e^{-\mathrm{i}\theta}\overline{t_j}|^2}$$

and

$$\lambda_2(H_\theta(A_j)) = \frac{1}{2}\mathrm{Re}(e^{\mathrm{i}\theta}(a_1+a_2)) - \frac{1}{2}\sqrt{(\mathrm{Re}(e^{\mathrm{i}\theta}(a_1-a_2)))^2 + |e^{\mathrm{i}\theta}s_j + e^{-\mathrm{i}\theta}\overline{t_j}|^2}.$$

Comparing the above eigenvalues, we have

$$\lambda_k(H_\theta(A)) \in \begin{cases} \{\lambda_1(H_\theta(A_1)), \ldots, \lambda_1(H_\theta(A_r))\}, & k \le r \\ \left\{\mathrm{Re}(e^{\mathrm{i}\theta}c)\right\}, & r < k \le n-r \\ \{\lambda_2(H_\theta(A_1)), \ldots, \lambda_2(H_\theta(A_r))\}, & n-r < k \le n. \end{cases}$$

At this point, we notice that for all $\theta \in [0, 2\pi)$ and by a suitable permutation π_θ of the integers $1, \ldots, r$ the above eigenvalues are ordered non-increasingly as follows

$$\lambda_1(H_\theta(A_{\pi_\theta(1)})) \ge \cdots \ge \lambda_1(H_\theta(A_{\pi_\theta(r)})) \ge \lambda_2(H_\theta(A_{\pi_\theta(r)})) \ge \cdots \ge \lambda_2(H_\theta(A_{\pi_\theta(1)})).$$

Distinguishing among the values of k, we obtain the following cases:

i. Assume $1 \le k \le r$. If we consider the $r \times r$ diagonal hermitian matrix $H_r(\theta) = \mathrm{diag}\left[\lambda_1(H_\theta(A_1)), \ldots, \lambda_1(H_\theta(A_r))\right]$ and any of its $(r-k+1) \times (r-k+1)$ principal submatrices $H_{r-k+1}(\theta)$, K. Fan and G. Pall's generalized interlacing inequalities for hermitian matrices [12] imply

$$\lambda_k(H_r(\theta)) \le \lambda_1(H_{r-k+1}(\theta)) \le \lambda_1(H_r(\theta)),$$

for any $\theta \in [0, 2\pi)$. Now, considering all $\binom{r}{k-1}$ principal submatrices $H_{r-k+1}(\theta)$ of $H_r(\theta)$, we have

$$\lambda_k(H_r(\theta)) \le \max_{1 \le j_1 < \cdots < j_{r-k+1} \le r} \{\lambda_1(H_\theta(A_{j_1})), \ldots, \lambda_1(H_\theta(A_{j_{r-k+1}}))\},$$

for any $\theta \in [0, 2\pi)$. Clearly, for any $\theta \in [0, 2\pi)$

$$\lambda_k(H_\theta(A)) = \min \max_{1 \le j_1 < \cdots < j_{r-k+1} \le r} \{\lambda_1(H_\theta(A_{j_1})), \ldots, \lambda_1(H_\theta(A_{j_{r-k+1}}))\}.$$

Hence, according to (1.1)

$$\Lambda_k(A) = \bigcap_{1 \le j_1 < \cdots < j_{r-k+1} \le r} \Lambda_1(A_{j_1} \oplus \cdots \oplus A_{j_{r-k+1}})$$

$$= \bigcap_{1\le j_1<\cdots<j_{r-k+1}\le r} \mathrm{conv}\left(\Lambda_1(A_{j_1})\cup\cdots\cup\Lambda_1(A_{j_{r-k+1}})\right).$$

The Elliptical Range Theorem in [15] establishes that $\Lambda_1(A_j)$, $j=1,\dots,r$ are elliptical disks all centered at $\frac{a_1+a_2}{2}$. Apparently, the convex hulls of their union have a nonempty intersection over all $r-k+1$ selections from the collection of r numerical ranges $\Lambda_1(A_j)$, thus ensuring $\Lambda_k(A)$ to be nonempty.

ii. Assume $r+1\le k\le n-r$. Then $\lambda_k(H_\theta(A))=\mathrm{Re}(e^{i\theta}c)$ and by (1.1), $\Lambda_k(A)=\{c\}$.

iii. Assume $n-r+1\le k\le n$. Then $\lambda_k(H_\theta(A))\le\lambda_2(H_\theta(A_j))$, for any $\theta\in[0,2\pi)$ and any index $j\in\{1,\dots,r\}$. By (1.1), we have $\Lambda_k(A)\subseteq\Lambda_2(A_j)=\emptyset$, since A_j is not a scalar matrix.

In case $r=n/2$, we have $A=A_1\oplus\cdots\oplus A_r$ and

$$\lambda_k(H_\theta(A))\in\begin{cases}\{\lambda_1(H_\theta(A_1)),\dots,\lambda_1(H_\theta(A_r))\}, & k\le r\\ \{\lambda_2(H_\theta(A_1)),\dots,\lambda_2(H_\theta(A_r))\}, & r<k\le n.\end{cases}$$

Analogously to the discussion above, we derive the second assertion. $\square$

Example 2. Let $A\in\mathcal{M}_{10}(\mathbb{C})$ be the direct sum of the matrices

$$A_1=\begin{bmatrix}2 & 6\\ -8-2i & -\sqrt{3}\end{bmatrix}, A_2=\begin{bmatrix}2 & 4.2\\ -i & -\sqrt{3}\end{bmatrix}, A_3=\begin{bmatrix}2 & 8\\ 2-8i & -\sqrt{3}\end{bmatrix},$$
$$A_4=\begin{bmatrix}2 & 2.1\\ -5+i & -\sqrt{3}\end{bmatrix} \text{ and } A_5=\begin{bmatrix}2 & 1\\ 7 & -\sqrt{3}\end{bmatrix}.$$

The set $\Lambda_1(A)$ coincides with the convex hull of the union $F(A_1)\cup\cdots\cup F(A_5)$, hence we shall demonstrate the validity of Theorem 1 for $\Lambda_k(A)$, $k=2,3,4,5$, respectively. Figs. 1(a), 1(c), 1(e), 1(g) depict the boundaries of numerical ranges of direct sums $\oplus_\gamma A_\gamma$ over all index sets $\gamma\subseteq\{1,\dots,5\}$ of cardinality $6-k$ for $k=2,3,4,5$, respectively. On the other hand, Figs. 1(b), 1(d), 1(f) and 1(h) illustrate the boundary and interior (white area) of $\Lambda_k(A)$ for $k=2,3,4,5$, respectively, by using the formula (1.1) for 200 tangent lines. Comparing our pictures, the corresponding sets are equal, thus confirming Theorem 1.

The following statement constitutes part of the proof of Theorem 2.1 in [4]. We state it here as a separate lemma for convenience of reference.

Lemma 3. *Let $X\in\mathcal{M}_{m,n}(\mathbb{C})$ and $Y\in\mathcal{M}_{n,m}(\mathbb{C})$ be such that XY and YX are normal matrices. Then for some unitary matrix U we have*

$$U^*\begin{bmatrix}a_1I_m & X\\ Y & a_2I_n\end{bmatrix}U=\begin{cases}a_1I_{m-n}\bigoplus_{i=1}^n A_j, & \textit{if}\quad m>n,\\ \qquad \textit{or} & \\ a_2I_{n-m}\bigoplus_{i=1}^m A_j, & \textit{if}\quad m<n,\end{cases}$$

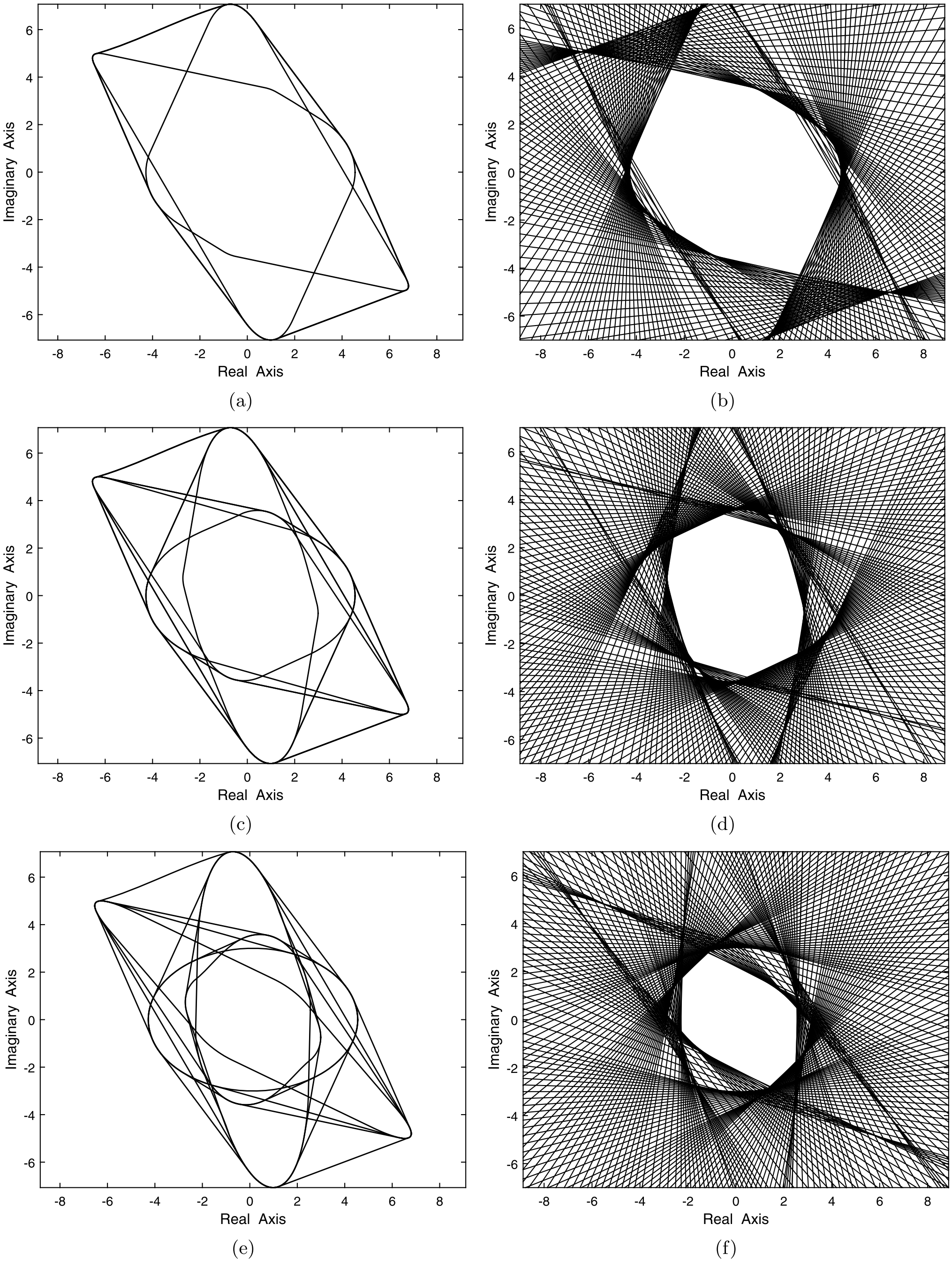
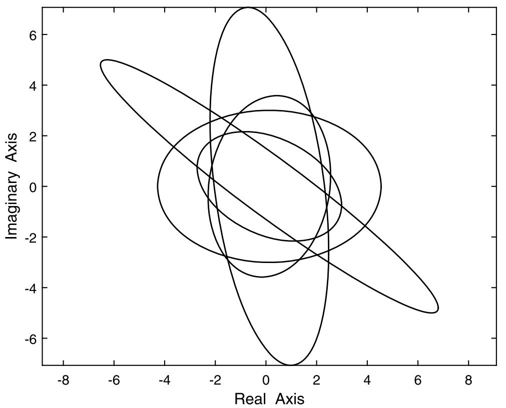
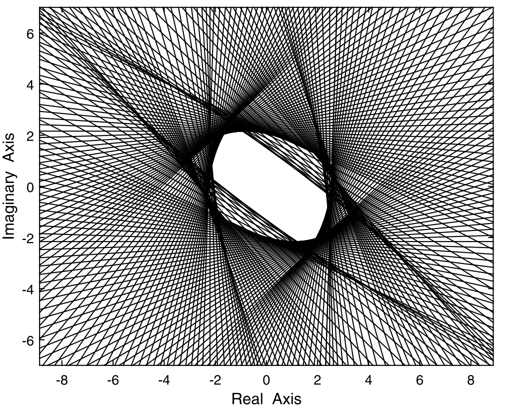


Fig. 1. The left figures illustrate the boundaries of $F(\oplus_\gamma A_\gamma)$ over all index sets $\gamma \subseteq \{1, \dots, 5\}$ of cardinality $6-k$, for $k = 2, 3, 4, 5$, respectively. The corresponding intersections give $\Lambda_k(A)$ at the right side for $k = 2, 3, 4, 5$, respectively.



(g)



(h)

Fig. 1. (*continued*)

where $A_j = \begin{bmatrix} a_1 & s_j \\ t_j & a_2 \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})$, $s_j^2 \in \sigma(XX^*)$ *and* $|t_j|^2 \in \sigma(YY^*)$, *for* $j = 1, \ldots, \min\{m, n\}$.

From Lemma 3 and Theorem 1 we immediately obtain

Proposition 4. *Let* $A \in \mathcal{M}_n(\mathbb{C})$ *be unitarily equivalent to the matrix*

$$\begin{bmatrix} a_1 I_p & X \\ Y & a_2 I_q \end{bmatrix} \in \mathcal{M}_n(\mathbb{C}), \tag{2.2}$$

with XY *and* YX *normal matrices. If* $p \neq q$, *then*

$$\Lambda_k(A) = \begin{cases} \bigcap_{1 \leq j_1 < \cdots < j_{r-k+1} \leq r} \operatorname{conv}\left(F(A_{j_1}) \cup \cdots \cup F(A_{j_{r-k+1}})\right), & \text{if } k \leq r := \min\{p, q\}, \\ \{a_1\} \ \text{or} \ \{a_2\}, & \text{if } r < k \leq n - r, \\ \emptyset, & \text{otherwise}, \end{cases}$$

where $A_j = \begin{bmatrix} a_1 & s_j \\ t_j & a_2 \end{bmatrix}$, *with* $s_j^2 \in \sigma(XX^*)$, $|t_j|^2 \in \sigma(YY^*)$, $j = 1, \ldots, r$.

If $p = q := r$, *then*

$$\Lambda_k(A) = \begin{cases} \bigcap_{1 \leq j_1 < \cdots < j_{r-k+1} \leq r} \operatorname{conv}\left(F(A_{j_1}) \cup \cdots \cup F(A_{j_{r-k+1}})\right), & \text{if } k \leq r, \\ \emptyset, & \text{if } r < k \leq n. \end{cases}$$

Proof. Due to the unitary equivalence property (P_2) of the rank-k numerical range, we may assume that A has already the form (2.2). Then the result is an immediate consequence of Theorem 1 and Lemma 3. $\square$

3. 2-Toeplitz matrices

A matrix is *tridiagonal* if its non-zero entries are all located on its main diagonal, the first diagonal below, and the first diagonal above the main one. A matrix is *Toeplitz* if its elements are constant along each descending diagonal. Therefore, matrices elements of which exhibit m-periodic behavior along diagonals are sometimes called m-*Toeplitz*.

In this section, we consider tridiagonal 2-Toeplitz matrices

$$T_n(b_1, b_2; a_1, a_2; c_1, c_2) = \begin{bmatrix} a_1 & c_1 & 0 & & \cdots & 0 \\ b_1 & a_2 & c_2 & \ddots & & \\ 0 & b_2 & a_1 & c_1 & & \vdots \\ & & b_1 & a_2 & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \\ 0 & \cdots & 0 & & \ddots & \ddots \end{bmatrix}. \tag{3.1}$$

In case of tridiagonal truly Toeplitz $n \times n$ matrix this notation will be abbreviated to $T_n(b, a, c)$. We also refer to the rectangular $m \times n$ tridiagonal Toeplitz matrix $T_{m,n}(b, a, c)$ as it will be used in the discussion below.

Our study focuses on the rank-k numerical range of 2-Toeplitz matrices (3.1) under an additional condition on their off-diagonal elements. This condition is sufficient for the rank-k numerical range to be of an elliptical shape. Before stating the corresponding result, we formulate (and prove) an auxiliary lemma.

Lemma 5. *Let the $n \times n$ 2-Toeplitz matrix $T_n(b_1, b_2; a_1, a_2; c_1, c_2)$ as in (3.1) be such that*

$$\overline{b_1}/c_1 = \overline{c_2}/b_2 := \mu. \tag{3.2}$$

Then there exists a unitary matrix U satisfying

$$U^* T_n(b_1, b_2; a_1, a_2; c_1, c_2) U = \begin{cases} A_1 \oplus \cdots \oplus A_\rho, & \textit{if } \ n = 2\rho, \ \rho \in \mathbb{N}, \\ \quad \textit{or} & \\ a_1 I_1 \oplus A_1 \oplus \cdots \oplus A_\rho, & \textit{if } \ n = 2\rho + 1, \ \rho \in \mathbb{N}, \end{cases}$$

where $A_j = \begin{bmatrix} a_1 & s_j \\ \overline{\mu} s_j & a_2 \end{bmatrix}$, $j = 1, \ldots, \rho$ and s_j is the j-th largest singular value of either $T_\rho(b_2, c_1, 0)$, if $n = 2\rho$ or $T_{\rho+1,\rho}(b_2, c_1, 0)$, if $n = 2\rho + 1$.

Proof. Let $n = 2\rho$, $\rho \in \mathbb{N}$. We consider the $n \times n$ permutation matrix $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$, with the $\rho \times n$ matrices $P_1 = \begin{bmatrix} e_1 & e_3 & \ldots & e_{2\rho-1} \end{bmatrix}^T$ and $P_2 = \begin{bmatrix} e_2 & e_4 & \ldots & e_{2\rho} \end{bmatrix}^T$, where

$e_j \in \mathbb{C}^n$ is the j-th (column) vector of the standard basis of $\mathbb{C}^n$. Applying P to the rows and columns of the matrix $T_n(b_1, b_2; a_1, a_2; c_1, c_2)$ given by (3.1), we have

$$PT_n(b_1, b_2; a_1, a_2; c_1, c_2)P^T = \begin{bmatrix} a_1 I_\rho & X \\ Y & a_2 I_\rho \end{bmatrix},$$

where $X = T_\rho(b_2, c_1, 0)$ and $Y = T_\rho(0, b_1, c_2)$. From (3.2) it follows that $Y = \overline{\mu} X^*$. The matrices $XY = \overline{\mu} XX^*$ and $YX = \overline{\mu} X^* X$ differ from positive semi-definite by a scalar multiple only, and thus are normal. By Proposition 4 there is a unitary matrix U such that

$$U^* \begin{bmatrix} a_1 I_\rho & X \\ Y & a_2 I_\rho \end{bmatrix} U = A_1 \oplus \cdots \oplus A_\rho,$$

where $A_j = \begin{bmatrix} a_1 & s_j \\ \overline{\mu} s_j & a_2 \end{bmatrix}$, $j = 1, \ldots, \rho$ with $s_1 \geq s_2 \geq \cdots \geq s_\rho \geq 0$ the singular values of X.

The case when n is odd, i.e. $n = 2\rho + 1$, is treated in a similar setting by taking the $(\rho + 1) \times n$ matrix $P_1 = \begin{bmatrix} e_1 & e_3 & \cdots & e_{2\rho-1} & e_{2\rho} \end{bmatrix}^T$ and the $\rho \times n$ matrix P_2. Hence,

$$PT_n(b_1, b_2; a_1, a_2; c_1, c_2)P^T = \begin{bmatrix} a_1 I_{\rho+1} & Z \\ W & a_2 I_\rho \end{bmatrix},$$

where $Z = T_{\rho+1,\rho}(b_2, c_1, 0)$ and $W = \overline{\mu} Z^*$. In this case, for some unitary matrix U, Proposition 4 yields

$$U^* \begin{bmatrix} a_1 I_{\rho+1} & Z \\ W & a_2 I_\rho \end{bmatrix} U = a_1 I_1 \oplus A_1 \oplus \cdots \oplus A_\rho,$$

where $A_j = \begin{bmatrix} a_1 & s_j \\ \overline{\mu} s_j & a_2 \end{bmatrix}$, $j = 1, \ldots, \rho$ with $s_1 \geq s_2 \geq \cdots \geq s_\rho \geq 0$ the singular values of Z. $\square$

Now, Lemma 5 and Proposition 4 imply the next theorem, which characterizes exactly the rank-k numerical range of a special tridiagonal 2-Toeplitz matrix.

Theorem 6. *Let the $n \times n$ 2-Toeplitz matrix $T_n(b_1, b_2; a_1, a_2; c_1, c_2)$ given by (3.1) be such that $\overline{b_1} b_2 = c_1 \overline{c_2}$. If $k \leq n/2$, then $\Lambda_k(T_n(b_1, b_2; a_1, a_2; c_1, c_2))$ is an elliptical disk centered at $(a_1 + a_2)/2$, with major axis of length*

$$L = \left(\frac{|a_1 - a_2|^2}{2} + \sum_{j=1}^{2}(|b_j|^2 + |c_j|^2) + 2 \sum_{i,j=1, i\neq j}^{2} |b_i c_j| \cos\left(\frac{2k\pi}{n+1}\right) + \right.$$
$$\left. + \left| \frac{(a_1 - a_2)^2}{2} + 2\sum_{j=1}^{2} b_j c_j + 4\left|\frac{b_2}{c_1}\right| b_1 c_1 \cos\left(\frac{2k\pi}{n+1}\right) \right| \right)^{1/2}$$

and minor axis of length

$$l = \left(\frac{|a_1 - a_2|^2}{2} + \sum_{j=1}^{2}(|b_j|^2 + |c_j|^2) + 2 \sum_{i,j=1, i\neq j}^{2} |b_i c_j| \cos\left(\frac{2k\pi}{n+1}\right) - \right.$$
$$\left. - \left| \frac{(a_1 - a_2)^2}{2} + 2\sum_{j=1}^{2} b_j c_j + 4\left|\frac{b_2}{c_1}\right| b_1 c_1 \cos\left(\frac{2k\pi}{n+1}\right) \right| \right)^{1/2}.$$

The major axis of the elliptical disk is parallel to the direction of the vector $e^{i\phi}$, *where*

$$\phi = (\arg d)/2 \text{ for } d = \frac{(a_1 - a_2)^2}{4} + \sum_{j=1}^{2} b_j c_j + \frac{2|b_2| b_1 c_1}{|c_1|} \cos\left(\frac{2k\pi}{n+1}\right).$$

If $n = 2\rho + 1$, $\rho \in \mathbb{N}$, *then* $\Lambda_{\rho+1}(T_n(b_1, b_2; a_1, a_2; c_1, c_2)) = \{a_1\}$. *For all other values of* k, *the rank-*k *numerical range is the empty set.*

Proof. Let $n = 2\rho$, $\rho \in \mathbb{N}$. The first cases of Lemma 5 and Proposition 4 imply that

$$\Lambda_k(T_n(b_1, b_2; a_1, a_2; c_1, c_2)) = \begin{cases} \bigcap_{1\leq j_1<\cdots<j_{\rho-k+1}\leq\rho} F(A_{j_1} \oplus \cdots \oplus A_{j_{\rho-k+1}}), & \text{if } k \leq \rho, \\ \emptyset, & \text{otherwise,} \end{cases} \tag{3.3}$$

where $A_j = \begin{bmatrix} a_1 & s_j \\ \overline{\mu} s_j & a_2 \end{bmatrix}$ $(j = 1, \ldots, \rho)$, $\mu = \frac{\overline{b_1}}{c_1} = \frac{\overline{c_2}}{b_2}$ and $s_1 \geq s_2 \geq \cdots \geq s_\rho \geq 0$ are the singular values of $X = T_\rho(b_2, c_1, 0)$. As argued in the proof of [4, Corollary 2.3],

$$F(A_1) \supseteq F(A_2) \supseteq \cdots \supseteq F(A_\rho) \tag{3.4}$$

are elliptical disks centered at $\frac{a_1+a_2}{2}$ with major and minor axes of length

$$L_j = \left(\frac{|a_1 - a_2|^2}{2} + (1 + |\mu|^2)s_j^2 + 2\left| \frac{(a_1 - a_2)^2}{4} + \overline{\mu} s_j^2 \right| \right)^{1/2}$$

and

$$l_j = \left(\frac{|a_1 - a_2|^2}{2} + (1 + |\mu|^2)s_j^2 - 2\left| \frac{(a_1 - a_2)^2}{4} + \overline{\mu} s_j^2 \right| \right)^{1/2},$$

respectively, for $j = 1, \ldots, \rho$. If $k \leq \rho$, the formula (3.3) together with (3.4) imply

$$\Lambda_k(T_n(b_1, b_2; a_1, a_2; c_1, c_2)) = F(A_1) \cap F(A_2) \cap \cdots \cap F(A_k) = F(A_k).$$

Now, let us compute the k-th largest singular value s_k of X. This is possible due to the structure of the $\rho \times \rho$ tridiagonal matrix

$$X^*X = \begin{bmatrix} |b_2|^2 + |c_1|^2 & \overline{b_2}c_1 & 0 & 0 \\ b_2\overline{c_1} & \ddots & \ddots & 0 \\ 0 & \ddots & |b_2|^2 + |c_1|^2 & \overline{b_2}c_1 \\ 0 & 0 & b_2\overline{c_1} & |c_1|^2 \end{bmatrix}.$$

Indeed, the matrix X^*X is nothing but the tridiagonal Toeplitz matrix $T_\rho(b_2\overline{c_1}, |b_2|^2 + |c_1|^2, \overline{b_2}c_1)$ with $|b_2|^2$ subtracted from its (ρ, ρ) entry.

The eigenvalues of such matrices have been analytically calculated in [13,18], whereupon

$$s_k^2 = |b_2|^2 + |c_1|^2 + 2|b_2c_1| \cos\left(\frac{2k\pi}{n+1}\right).$$

Hence, our assertion is readily verified by the Elliptical Range Theorem [15].

The case $n = 2\rho + 1$, $\rho \in \mathbb{N}$ is treated in a similar setting by taking the second case of Lemma 5. Then Proposition 4 yields

$$\Lambda_k(T_n(b_1, b_2; a_1, a_2; c_1, c_2)) = \begin{cases} \bigcap_{1 \leq j_1 < \cdots < j_{\rho-k+1} \leq \rho} F(A_{j_1} \oplus \cdots \oplus A_{j_{\rho-k+1}}), & k \leq \rho, \\ \{a_1\}, & k = \rho + 1, \\ \emptyset, & \text{otherwise}, \end{cases}$$

where $A_j = \begin{bmatrix} a_1 & s_j \\ \overline{\mu} s_j & a_2 \end{bmatrix}$, $j = 1, \ldots, \rho$ and $s_j \geq 0$ are the singular values of $Z = T_{\rho+1,\rho}(b_2, c_1, 0)$ arranged in decreasing order. In the context of the previous arguments for even order n, we also obtain $\Lambda_k(T_n(b_1, b_2; a_1, a_2; c_1, c_2)) = F(A_k)$, when $k \leq \rho$. Then the k-th largest eigenvalue s_k of the $\rho \times \rho$ tridiagonal Toeplitz matrix

$$Z^*Z = T_\rho(b_2\overline{c_1}, |b_2|^2 + |c_1|^2, \overline{b_2}c_1)$$

is determined by $s_k^2 = |b_2|^2 + |c_1|^2 + 2|b_2c_1| \cos\left(\frac{2k\pi}{n+1}\right)$ [13,18]. □

The significance of the conditions imposed in Theorem 6 is illustrated by the following example showing that the rank-k numerical range of an arbitrary tridiagonal 2-Toeplitz matrix is not always an elliptical disk.

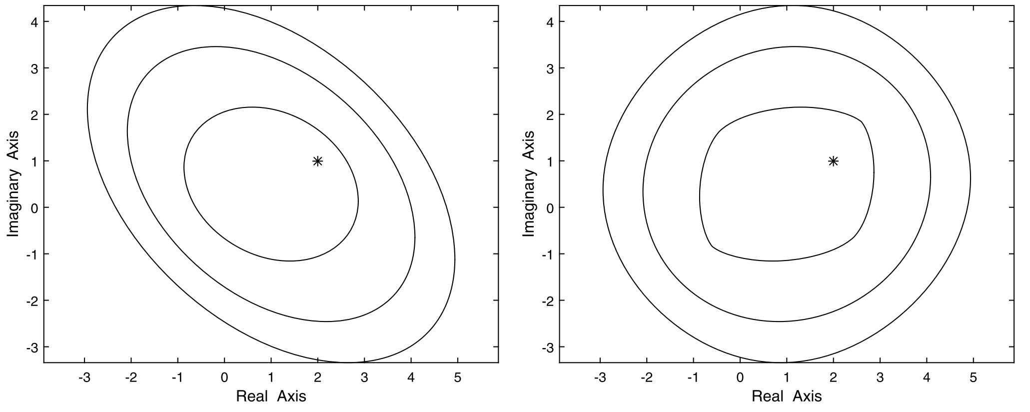


Fig. 2. Fig. 2(a) depicts $\Lambda_k(T_7(-\mathrm{i}, -4\mathrm{i}; 2+\mathrm{i}, 0; 4, 1))$ for $k = 1, \ldots, 3$ from outside inside, respectively and Fig. 2(b) depicts $\Lambda_k(T_7(\mathrm{i}, -4\mathrm{i}; 2+\mathrm{i}, 0; 4, 1))$ for $k = 1, \ldots, 3$. Furthermore, $\Lambda_4(T_7(-\mathrm{i}, -4\mathrm{i}; 2+\mathrm{i}, 0; 4, 1)) = \Lambda_4(T_7(\mathrm{i}, -4\mathrm{i}; 2+\mathrm{i}, 0; 4, 1)) = \{2+\mathrm{i}\}$.

Example 7. We take the 7×7 2-Toeplitz matrix as in (3.1) with $\overline{b_1}b_2 = c_1\overline{c_2} = 4$, that is

$$T_7(-\mathrm{i}, -4\mathrm{i}; 2+\mathrm{i}, 0; 4, 1) = \begin{bmatrix} 2+\mathrm{i} & 4 & 0 & 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -4\mathrm{i} & 2+\mathrm{i} & 4 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4\mathrm{i} & 2+\mathrm{i} & 4 & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{i} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -4\mathrm{i} & 2+\mathrm{i} \end{bmatrix}.$$

The elliptic curves in Fig. 2(a) depict the boundaries of $\Lambda_1(T_7(-\mathrm{i}, -4\mathrm{i}; 2+\mathrm{i}, 0; 4, 1)) \supseteq \Lambda_2(T_7(-\mathrm{i}, -4\mathrm{i}; 2+\mathrm{i}, 0; 4, 1)) \supseteq \Lambda_3(T_7(-\mathrm{i}, -4\mathrm{i}; 2+\mathrm{i}, 0; 4, 1))$ from outside inside, respectively. Notice that $\Lambda_4(T_7(-\mathrm{i}, -4\mathrm{i}; 2+\mathrm{i}, 0; 4, 1)) = \{2+\mathrm{i}\}$ is marked with a "star" and $\Lambda_k(T_7(-\mathrm{i}, -4\mathrm{i}; 2+\mathrm{i}, 0; 4, 1)) = \emptyset$, for $k = 5, 6$.

If $b_1 = \mathrm{i}$, then the condition $\overline{b_1}b_2 = c_1\overline{c_2}$ does not hold and $\Lambda_k(T_7(\mathrm{i}, -4\mathrm{i}; 2+\mathrm{i}, 0; 4, 1))$ fail to be elliptical disks for all k, as illustrated in Fig. 2(b).

We further restrict our previous result to an $n \times n$ *continuant* matrix $T_n(b_1, b_2; a_1, a_2; -\overline{b_1}, -\overline{b_2})$ [5,10]. This terminology comes from the well known relation between ratios of the determinant of nested continuant matrices and a continued fraction involving its nonzero entries:

$$\frac{\det T_n(b_1, b_2; a_1, a_2; -\overline{b_1}, -\overline{b_2})}{\det T_{n-1}(b_2, b_1; a_2, a_1; -\overline{b_2}, -\overline{b_1})} = a_1 + \cfrac{|b_1|^2}{a_2 + \cfrac{|b_2|^2}{a_1 + \frac{|b_1|^2}{a_2 + \cdots}}}.$$

Proposition 8. *Let $T_n(b_1, b_2; a_1, a_2; -\overline{b_1}, -\overline{b_2})$ be an $n \times n$ complex continuant matrix. If $k \le n/2$, then $\Lambda_k(T_n(b_1, b_2; a_1, a_2; -\overline{b_1}, -\overline{b_2}))$ is an elliptical disk centered at $(a_1 + a_2)/2$, major axis parallel to the vector $e^{i\phi}$ of length*

$$L = \left(\frac{|a_1 - a_2|^2}{2} + 2(|b_1| - |b_2|)^2 + 8\,|b_1 b_2| \cos^2\left(\frac{k\pi}{n+1}\right) \right. \\ \left. + \left| \frac{(a_1 - a_2)^2}{2} - 2(|b_1| - |b_2|)^2 - 8\,|b_1 b_2| \cos^2\left(\frac{k\pi}{n+1}\right) \right| \right)^{1/2}$$

and minor axis of length

$$l = \left(\frac{|a_1 - a_2|^2}{2} + 2(|b_1| - |b_2|)^2 + 8\,|b_1 b_2| \cos^2\left(\frac{k\pi}{n+1}\right) \right. \\ \left. - \left| \frac{(a_1 - a_2)^2}{2} - 2(|b_1| - |b_2|)^2 - 8\,|b_1 b_2| \cos^2\left(\frac{k\pi}{n+1}\right) \right| \right)^{1/2},$$

where

$$\frac{(a_1 - a_2)^2}{2} - 2(|b_1| - |b_2|)^2 - 8\,|b_1 b_2| \cos^2\left(\frac{k\pi}{n+1}\right) \\ = e^{2i\phi} \left| \frac{(a_1 - a_2)^2}{2} - 2(|b_1| - |b_2|)^2 - 8\,|b_1 b_2| \cos^2\left(\frac{k\pi}{n+1}\right) \right|.$$

If $n = 2\rho + 1$, $\rho \in \mathbb{N}$, then $\Lambda_{\rho+1}(T_n(b_1, b_2; a_1, a_2; -\overline{b_1}, -\overline{b_2})) = \{a_1\}$. For all other values of k, the set is empty.

Proof. It is an immediate consequence of Theorem 6, replacing $c_1 = -\overline{b_1}$, $c_2 = -\overline{b_2}$ and taking into consideration the trigonometric identity

$$\cos\left(\frac{2k\pi}{n+1}\right) = 2\cos^2\left(\frac{k\pi}{n+1}\right) - 1. \quad \square$$

Proposition 9. *Let $T_n(b_1, b_2; a_1, a_2; -b_1, -b_2)$ be an $n \times n$ real continuant matrix. If $k \le n/2$, then $\Lambda_k(T_n(b_1, b_2; a_1, a_2; -b_1, -b_2))$ is an elliptical disk centered at $(a_1 + a_2)/2$, with horizontal axis of length $|a_1 - a_2|$ and vertical axis of length $2\left[(b_1 - b_2)^2 + 4|b_1 b_2| \cos^2\left(\frac{k\pi}{n+1}\right)\right]^{1/2}$.*

If $n = 2\rho+1$, $\rho \in \mathbb{Z}$, then $\Lambda_{\rho+1}(T_n(b_1, b_2; a_1, a_2; -b_1, -b_2))$ degenerates to the singleton $\{a_1\}$. All other values of k give an empty set.

Proof. Proposition 8 infers $\Lambda_k(T_n(b_1, b_2; a_1, a_2; -b_1, -b_2))$ to be an elliptical disk centered at $(a_1 + a_2)/2$. For its horizontal axis ($\phi = 0$) we have

$$\left|\frac{(a_1-a_2)^2}{2} - 2(b_1-b_2)^2 - 8\,|b_1b_2|\cos^2\left(\frac{k\pi}{n+1}\right)\right|$$
$$= \frac{(a_1-a_2)^2}{2} - 2(b_1-b_2)^2 - 8\,|b_1b_2|\cos^2\left(\frac{k\pi}{n+1}\right).$$

Hence, $L = |a_1 - a_2|$ is its horizontal axis length, while $l = 2\big[(b_1 - b_2)^2 + 4|b_1b_2|\cos^2\big(\frac{k\pi}{n+1}\big)\big]^{1/2}$ is its vertical axis length. □

The implications of Theorem 6 are also the same if we perform an interchange between the non diagonal elements b_1, c_1 or b_2, c_2 of $T_n(b_1, b_2; a_1, a_2; c_1, c_2)$. In order to prove this statement, we need the following auxiliary lemma.

Lemma 10. *The spectrum of any $n \times n$ tridiagonal matrix is invariant under interchange of its $(j, j+1)$ and $(j+1, j)$ entries for any $j = 1, \dots, n-1$.*

Proof. Consider $n \times n$ tridiagonal matrices

$$A_n = \begin{bmatrix} a_1 & c_1 & 0 & & \cdots & 0 \\ b_1 & \ddots & \ddots & & & \\ 0 & \ddots & a_j & c_j & \ddots & \vdots \\ & & b_j & a_{j+1} & & \\ \vdots & & & & \ddots & 0 \\ & & & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & & 0 & b_{n-1} & a_n \end{bmatrix} \quad \text{and}$$

$$\tilde{A}_n = \begin{bmatrix} a_1 & c_1 & 0 & & \cdots & 0 \\ b_1 & \ddots & \ddots & & & \\ 0 & \ddots & a_j & b_j & \ddots & \vdots \\ & & c_j & a_{j+1} & & \\ \vdots & & & & \ddots & 0 \\ & & & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & & 0 & b_{n-1} & a_n \end{bmatrix},$$

the latter being obtained from the former by the interchange of its $(j, j+1)$ and $(j+1, j)$ entries. Denote their characteristic polynomials by $p_n(x) = \det(A_n - xI_n)$ and $\widetilde{p}_n(x) = \det(\tilde{A}_n - xI_n)$, respectively. There is a well known three term recurrence formula

$$\widetilde{p}_n(x) = a_n\widetilde{p}_{n-1}(x) - b_{n-1}c_{n-1}\widetilde{p}_{n-2}(x), \tag{3.5}$$

where $n \geq 2$ and $\widetilde{p}_0(x) = 1$ [6, Lemma 1]. For $n = j+1$, we have

$$\begin{aligned}\widetilde{p}_{j+1}(x) &= a_{j+1}\widetilde{p}_j(x) - b_j c_j \widetilde{p}_{j-1}(x)\\ &= a_{j+1}p_j(x) - c_j b_j p_{j-1}(x) = p_{j+1}(x).\end{aligned}$$

Clearly, applying the above equality in the recursion (3.5), we derive that $\widetilde{p}_n(x) = p_n(x)$. □

Theorem 11. *Let the $n\times n$ 2-Toeplitz matrix $T_n(b_1, b_2; a_1, a_2; c_1, c_2)$ given by (3.1) be such that $\overline{b_1}c_2 = c_1\overline{b_2}$. Then its rank-k numerical range is the same elliptical disk as described by Theorem 6.*

Proof. Taking into consideration (1.1), we have

$$\begin{aligned}&\Lambda_k(T_n(b_1, b_2; a_1, a_2; c_1, c_2))\\ &\quad= \bigcap_{\theta\in[0,2\pi)} e^{-\mathrm{i}\theta}\{z \in \mathbb{C} : \mathrm{Re}z \le \lambda_k(H_\theta(T_n(b_1, b_2; a_1, a_2; c_1, c_2)))\},\end{aligned}$$

where $T_n(b_1, b_2; a_1, a_2; c_1, c_2)$ is a 2-Toeplitz matrix such that $\overline{b_1}c_2 = c_1\overline{b_2}$.

We notice that $H_\theta(T_n(b_1, b_2; a_1, a_2; c_1, c_2))$ are tridiagonal matrices for any $\theta \in [0, 2\pi]$). Lemma 10 yields that their spectrum remains unchanged after performing an interchange of corresponding off-diagonal elements. This fact assures the equality of the k-th largest eigenvalues as follows:

$$\lambda_k(H_\theta(T_n(b_1, b_2; a_1, a_2; c_1, c_2))) = \lambda_k(H_\theta(T_n(b_1, c_2; a_1, a_2; c_1, b_2))),$$

which leads to the equation $\overline{b_1}b_2 = c_1\overline{c_2}$. Hence, Theorem 6 implies the result. □

4. Tridiagonal and s-tridiagonal Toeplitz matrices

It is well known [11, Corollary 4] that the classical numerical range of a tridiagonal Toeplitz matrix $T_n(b, a, c)$ coincides with an elliptical disk. Namely,

$$F(T_n(b, a, c)) = \left\{bz + c\overline{z} :\ z \in \mathcal{D}\left(0, \cos\left(\frac{\pi}{n+1}\right)\right)\right\} + \{a\},$$

where $\mathcal{D}\left(0, \cos\left(\frac{\pi}{n+1}\right)\right)$ denotes the circular disk centered at the origin and having radius $\cos\left(\frac{\pi}{n+1}\right)$.

The spectrum of a tridiagonal Toeplitz matrix has been also calculated explicitly (see [3, Theorem 2.4] and also [14,20])

$$\lambda_j(T_n(b, a, c)) = a + 2(bc)^{1/2}\cos\left(\frac{j\pi}{n+1}\right), \quad j = 1, 2, \ldots, n. \tag{4.1}$$

Clearly, $\lambda_j(T_n(b,a,c))$ are simple eigenvalues of $T_n(b,a,c)$ lying on the (complex) line segment

$$\left\{a+\gamma e^{\mathrm{i}\frac{\arg(b)+\arg(c)}{2}} \,:\, -2\sqrt{|bc|}\cos\left(\frac{\pi}{n+1}\right) \leq \gamma \leq 2\sqrt{|bc|}\cos\left(\frac{\pi}{n+1}\right)\right\},$$

and they are located symmetrically with respect to point a.

As it happens, the ellipticity property persists for the rank-k numerical range with $k>1$.

Theorem 12. *If $k \leq n/2$, then the rank-k numerical range of an $n\times n$ tridiagonal Toeplitz matrix $T_n(b,a,c)$ is an elliptical disk centered at a, with major axis of length*

$$L=\left[2(|b|^2+|c|^2)\left(1+\cos\left(\frac{2k\pi}{n+1}\right)\right)+4|bc|\left|1+\cos\left(\frac{2k\pi}{n+1}\right)\right|\right]^{1/2}$$

and minor axis of length

$$l=\left[2(|b|^2+|c|^2)\left(1+\cos\left(\frac{2k\pi}{n+1}\right)\right)-4|bc|\left|1+\cos\left(\frac{2k\pi}{n+1}\right)\right|\right]^{1/2}.$$

The major axis of the elliptical disk forms the angle $\arg(bc)/2$ *with the positive direction of the x-axis.*

If $n=2\rho+1$, $\rho\in\mathbb{N}$, then $\Lambda_{\rho+1}(T_n(b,a,c))=\{a\}$. For all other values of k, the set is empty.

Proof. Apparently, $T_n(b,a,c)=T_n(b,b;a,a;c,c)$. By Theorems 6 and 11, $\Lambda_k(T_n(b,a,c))=\Lambda_k(T_n(b,b;a,a;c,c))=\Lambda_k(T_n(b,c;a,a;c,b))$, and the result is immediate. □

Proposition 13. *Let $\lambda_1(T_n(b,a,c)),\ldots,\lambda_n(T_n(b,a,c))$ be the eigenvalues of $T_n(b,a,c)$ as in* (4.1)*. Then $\lambda_k(T_n(b,a,c))$ and $\lambda_{n-k+1}(T_n(b,a,c))$ are the foci of the elliptical disk $\Lambda_k(T_n(b,a,c))$ ($k\leq n/2$).*

Proof. Without loss of generality, we can consider the matrix $T_n=T_n(b,0,c)$. The foci of the elliptical disk $\Lambda_k(T_n)$ ($k\leq n/2$) with respect to its major and minor axis length calculated in Theorem 12 are given by

$$|f|^2=\frac{L^2-l^2}{4}=2\,|bc|\left(1+\cos\left(\frac{2k\pi}{n+1}\right)\right)=4\,|bc|\cos^2\left(\frac{k\pi}{n+1}\right).$$

Then (4.1) along with the fact that the major axis of the elliptical disk forms the angle $\arg(bc)/2$ with the positive direction of the x-axis, yield

$$f_1 = 2(bc)^{1/2} \cos\left(\frac{k\pi}{n+1}\right) = \lambda_k(T_n) \text{ and}$$
$$f_2 = -2(bc)^{1/2} \cos\left(\frac{k\pi}{n+1}\right) = \lambda_{n-k+1}(T_n). \quad \square$$

If we move the constant elements lying on the super and sub diagonals of an $n \times n$ tridiagonal Toeplitz matrix to the s-th diagonal above and below the main diagonal, respectively, the result is a so called *s-tridiagonal Toeplitz matrix* denoted by $T_n^{(s)}(b,a,c)$ $(1 \leq s \leq n-1)$. A straightforward example for $s = 2$ is

$$T_n^{(2)}(b,a,c) = \begin{bmatrix} a & 0 & c & 0 & \cdots & 0 \\ 0 & a & 0 & c & \ddots & \vdots \\ b & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & b & \ddots & \ddots & 0 & c \\ \vdots & \ddots & \ddots & 0 & a & 0 \\ 0 & \cdots & 0 & b & 0 & a \end{bmatrix}. \tag{4.2}$$

Notice that 1-tridiagonal Toeplitz matrix $T_n^{(1)}(b,a,c) = T_n(b,a,c)$ is in fact a standard tridiagonal Toeplitz matrix.

The next result shows that the rank-k numerical range of an s-tridiagonal Toeplitz matrix is yet another elliptical disk.

Theorem 14. *Let $T_n^{(s)}(b,a,c)$ be an s-tridiagonal Toeplitz matrix. Then*

$$\Lambda_k(T_n^{(s)}(b,a,c)) = \begin{cases} \Lambda_{\lceil \frac{k}{s} \rceil}(T_\rho(b,a,c)), & \text{if } n = \rho s,\ \rho \in \mathbb{N}, \\ \Lambda_{\lceil \frac{k}{s} \rceil}(T_{\rho+1}(b,a,c)), & \text{if } n = \rho s + r \text{ and} \\ & (j-1)s < k \leq (j-1)s + r,\ j = 1, \ldots, \rho+1, \\ \Lambda_{\lceil \frac{k}{s} \rceil}(T_\rho(b,a,c)), & \text{if } n = \rho s + r \text{ and} \\ & (j-1)s + r < k \leq js,\ j = 1, \ldots, \rho, \end{cases}$$

where $\lceil \frac{k}{s} \rceil$ is the least integer greater than or equal to $\frac{k}{s}$.

Proof. We will use the algorithm presented in [19] to represent the s-tridiagonal Toeplitz matrix $T_n^{(s)}(b,a,c)$ as permutationally similar to a direct sum of tridiagonal Toeplitz matrices.

Namely, for $S_n = \{1, 2, \ldots, n\}$ denote by

$$[r] = \{z \in S_n : z \equiv r \,(\mathrm{mod}\ s)\},$$

its congruence classes modulo s.

Clearly, $r \in \{0, 1, \ldots, s-1\}$ and if $n \in [r]$, there exists an integer ρ such that $n = \rho s + r$. It is readily verified that the cardinality of the classes $|[1]| = |[2]| = \cdots =$

$|[r]| = \rho + 1$ and $|[0]| = |[r+1]| = \cdots = |[s-1]| = \rho$. Following [19], we take the $n \times n$ permutation matrix

$$P = [e_s \quad e_{2s} \quad \cdots \quad e_1 \quad e_{s+1} \quad e_{2s+1} \quad \cdots \quad e_{s-1} \quad e_{2s-1} \quad \cdots],$$

where e_j is the j-th column vector of the standard basis. Hence, we have

$$P^T T_n^{(s)}(b,a,c)P = T_\rho \oplus \underbrace{T_{\rho+1} \oplus \cdots \oplus T_{\rho+1}}_{r\text{-}times} \oplus \underbrace{T_\rho \oplus \cdots \oplus T_\rho}_{(s-r-1)\text{-}times},$$

where $T_\rho = T_\rho(b,a,c)$. By the unitary invariance property (P_2) of the rank-k numerical range and the relation (1.1), we derive that

$$\begin{aligned}\Lambda_k(T_n^{(s)}(b,a,c)) &= \Lambda_k\left(\bigoplus_{i=1}^{r} T_{\rho+1} \bigoplus_{i=1}^{s-r} T_\rho\right) \\ &= \bigcap_{\theta\in[0,2\pi)} e^{-\mathrm{i}\theta}\left\{z \in \mathbb{C} : \mathrm{Re} z \le \lambda_k\left(\bigoplus_{i=1}^{r} H_\theta(T_{\rho+1}) \bigoplus_{i=1}^{s-r} H_\theta(T_\rho)\right)\right\}.\end{aligned} \tag{4.3}$$

Apparently, $\sigma\left(\bigoplus_{i=1}^{r} H_\theta(T_{\rho+1}) \bigoplus_{i=1}^{s-r} H_\theta(T_\rho)\right) = \bigcup_{i=1}^{r} \sigma(H_\theta(T_{\rho+1})) \bigcup_{i=1}^{s-r} \sigma(H_\theta(T_\rho))$. Notice also that the $\rho \times \rho$ hermitian matrix $H_\theta(T_\rho)$ is imbeddable in the $(\rho+1)\times(\rho+1)$ hermitian matrix $H_\theta(T_{\rho+1})$. By [12, Theorem 1] and also taking into account the corresponding multiplicities, we obtain

$$\begin{aligned}
&\underbrace{\lambda_1(H_\theta(T_{\rho+1})) = \cdots = \lambda_1(H_\theta(T_{\rho+1}))}_{r\text{-}times} \ge \\
&\underbrace{\lambda_1(H_\theta(T_\rho)) = \cdots = \lambda_1(H_\theta(T_\rho))}_{(s-r)\text{-}times} \ge \\
&\underbrace{\lambda_2(H_\theta(T_{\rho+1})) = \cdots = \lambda_2(H_\theta(T_{\rho+1}))}_{r\text{-}times} \ge \\
&\underbrace{\lambda_2(H_\theta(T_\rho)) = \cdots = \lambda_2(H_\theta(T_\rho))}_{(s-r)\text{-}times} \ge \\
&\vdots \ge \\
&\underbrace{\lambda_\rho(H_\theta(T_{\rho+1})) = \cdots = \lambda_\rho(H_\theta(T_{\rho+1}))}_{r\text{-}times} \ge \\
&\underbrace{\lambda_\rho(H_\theta(T_\rho)) = \cdots = \lambda_\rho(H_\theta(T_\rho))}_{(s-r)\text{-}times} \ge \underbrace{\lambda_{\rho+1}(H_\theta(T_{\rho+1})) = \cdots = \lambda_{\rho+1}(H_\theta(T_{\rho+1}))}_{r\text{-}times}.
\end{aligned}$$

The above (in)equalities imply that

$$\lambda_k(H_\theta(T_n^{(s)}(b,a,c))) = \begin{cases} \lambda_1(H_\theta(T_{\rho+1})), & 1 \leq k \leq r \\ \lambda_1(H_\theta(T_\rho)), & r < k \leq s \\ \lambda_2(H_\theta(T_{\rho+1})), & s < k \leq s+r \\ \lambda_2(H_\theta(T_\rho)), & s+r < k \leq 2s \\ \vdots & \vdots \\ \lambda_\rho(H_\theta(T_{\rho+1})), & (\rho-1)s < k \leq (\rho-1)s+r \\ \lambda_\rho(H_\theta(T_\rho)), & (\rho-1)s+r < k \leq \rho s \\ \lambda_{\rho+1}(H_\theta(T_{\rho+1})), & \rho s < k \leq \rho s + r. \end{cases}$$

Thus (4.3) results in

$$\Lambda_k(T_n^{(s)}(b,a,c)) = \begin{cases} \Lambda_{\lceil \frac{k}{s} \rceil}(T_\rho(b,a,c)), & \text{if } n = \rho s,\ \rho \in \mathbb{N}, \\ \Lambda_{\lceil \frac{k}{s} \rceil}(T_{\rho+1}(b,a,c)), & \text{if } n = \rho s + r \text{ and} \\ & (j-1)s < k \leq (j-1)s + r,\ j = 1, \dots, \rho+1, \\ \Lambda_{\lceil \frac{k}{s} \rceil}(T_\rho(b,a,c)), & \text{if } n = \rho s + r \text{ and} \\ & (j-1)s + r < k \leq js,\ j = 1, \dots, \rho. \end{cases}$$

Due to Theorem 12, $\Lambda_k(T_n^{(s)}(b,a,c))$ is an elliptical disk. $\square$

To illustrate, for the matrix given by (4.2), we have in particular:

$$\Lambda_k(T_n^{(2)}(b,a,c)) = \begin{cases} \Lambda_{\lceil \frac{k}{2} \rceil}(T_\rho(b,a,c)), & \text{if } n = 2\rho,\ \rho \in \mathbb{N}, \\ \Lambda_{\lceil \frac{k}{2} \rceil}(T_{\rho+1}(b,a,c)), & \text{if } n = 2\rho+1 \text{ and } k \text{ is odd}, \\ \Lambda_{\frac{k}{2}}(T_\rho(b,a,c)), & \text{if } n = 2\rho+1 \text{ and } k \text{ is even}. \end{cases}$$

References

[1] Aik Aretaki, J. Maroulas, The higher rank numerical range of nonnegative matrices, Cent. Eur. J. Math. 11 (3) (2013) 435–446.
[2] Aik Aretaki, J. Maroulas, The k-rank numerical radii, Ann. Funct. Anal. 3 (1) (2012) 100–108.
[3] A. Bottcher, S.M. Grudsky, Spectral Properties of Banded Toeplitz Matrices, SIAM Publications, Philadelphia, 2005.
[4] E.S. Brown, I. Spitkovsky, On matrices with elliptical numerical ranges, Linear Multilinear Algebra 52 (2004) 177–193.
[5] E.S. Brown, I. Spitkovsky, On flat portions on the boundary of the numerical range, Linear Algebra Appl. 390 (2004) 75–109.
[6] N.D. Cahil, J.R. D'Errico, J.P. Spence, Complex factorizations of the Fibonacci and Lucas numbers, Fibonacci Quart. 41 (2003) 13–19.
[7] M.D. Choi, M. Giesinger, J.A. Holbrook, D.W. Kribs, Geometry of higher-rank numerical ranges, Linear Multilinear Algebra 56 (1) (2008) 53–64.
[8] M.D. Choi, D.W. Kribs, K. Zyczkowski, Higher-rank numerical ranges and compression problems, Linear Algebra Appl. 418 (2006) 828–839.
[9] M.D. Choi, D.W. Kribs, K. Zyczkowski, Quantum error correcting codes from the compression formalism, Rep. Math. Phys. 58 (2006) 77–86.

[10] M.T. Chien, J.M. Huang, Numerical range of a continuant matrix, Appl. Math. Lett. 14 (2001) 213–216.
[11] M. Eiermann, Fields of values and iterative methods, Linear Algebra Appl. 180 (1993) 167–197.
[12] K. Fan, G. Pall, Imbedding conditions for Hermitian and normal matrices, Canad. J. Math. 9 (1957) 298–304.
[13] C.M. da Fonseca, The characteristic polynomial of some perturbed tridiagonal *k*-Toeplitz matrices, Appl. Math. Sci. 1 (2007) 59–67.
[14] U. Grenander, G. Szegö, Toeplitz Forms and Their Applications, University of California Press, Berkley and Los Angeles California, 1958.
[15] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.
[16] C.K. Li, Y.T. Poon, N.S. Sze, Condition for the higher rank numerical range to be non-empty, Linear Multilinear Algebra 57 (4) (2009) 365–368.
[17] C.K. Li, N.S. Sze, Canonical forms, higher rank numerical ranges, totally isotropic subspaces and matrix equations, Proc. Amer. Math. Soc. 136 (2008) 3013–3023.
[18] G. Lombardi, R. Rebaudo, Eigenvalues and eigenvectors of a special class of band matrices, Rend. Istit. Mat. Univ. Trieste, Int. J. Math. 20 (1988) 113–128.
[19] M. El-Mikkawy, T. Sogabe, Fast block diagonalization of k-tridiagonal matrices, Appl. Math. Comput. 218 (6) (2011) 2740–2743.
[20] S. Noschese, L. Pasquini, L. Reichel, Tridiagonal Toeplitz matrices: properties and novel applications, Numer. Linear Algebra Appl. 20 (2) (2013) 302–326.
[21] H.J. Woerdeman, The higher rank numerical range is convex, Linear Multilinear Algebra 56 (2007) 65–67.