Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(1): 701–716.
DOI:10.3934/math.2020047
Received: 01 August 2019
Accepted: 02 December 2019
Published: 19 December 2019

Research article

Some new bounds on the spectral radius of nonnegative matrices

Maria Adam*, **Dimitra Aggeli and Aikaterini Aretaki**

Department of Computer Science and Biomedical Informatics, University of Thessaly

* **Correspondence:** Email: madam@dib.uth.gr; Tel: +0302231066900; Fax: +0302231066939.

Abstract: In this paper, we determine some new bounds for the spectral radius of a nonnegative matrix with respect to a new defined quantity, which can be considered as an average of average 2-row sums. The new formulas extend previous results using the row sums and the average 2-row sums of a nonnegative matrix. We also characterize the equality cases of the bounds if the matrix is irreducible and we provide illustrative examples comparing with the existing bounds.

Keywords: nonnegative matrix; spectral radius; row sum; 2-row sum; average 2-row sum; signless Laplacian matrix
Mathematics Subject Classification: 15A18, 15A42, 05C50

1. Introduction

Let $\mathcal{M}_n(\mathbb{R})$ be the algebra of $n \times n$ real matrices. We refer to $A = (a_{ij})_{i,j=1}^n \in \mathcal{M}_n(\mathbb{R})$ as a *nonnegative* or a *positive* matrix, when each $a_{ij} \geq 0$ or $a_{ij} > 0$, respectively, denoted by writing $A \geq 0$ or $A > 0$. The matrix $A \in \mathcal{M}_n(\mathbb{R})$ is called *irreducible* if and only if $(I + A)^{n-1} > 0$. We also define the *spectral radius* of A by

$$\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ denotes the spectrum of A, that is, the set of eigenvalues of A.

The spectral radius of a nonnegative matrix has been studied extensively in the mathematical fields of dynamical systems and graph theory (see, e.g. [1–4, 6, 9–13]). For example, the topological entropy, one of the main invariants of a topological dynamical system telling us how chaotic the system is, can often be computed as a logarithm of the spectral radius of a certain nonnegative matrix [13]. Also, from the perspective of graph theory, several combinatorial properties of simple undirected graphs have been interpreted via the spectral radius of the signless Laplacian matrix derived accordingly from adjacency and degree matrices of a graph (see e.g. [4, 6, 9, 11] and the references therein).

The problem of bounding the largest eigenvalue in modulus of a nonnegative matrix has attracted the interest of many researchers [2, 4–11]. The celebrated Perron-Frobenius theory investigated the existence of positive eigenvalues of nonnegative matrices [5, 8]. In particular, Perron proved that $\rho(A)$ is a positive and simple eigenvalue of $A > 0$ [8, Theorem 8.2.11], and Frobenius generalized Perron's statement to nonnegative and irreducible matrices [8, Theorem 8.4.4]. Moreover, apart from the familiar power method, other numerical algorithms have been also constructed and implemented for locating the spectrum of matrices as in [3, 12]. However, the proposed methods are only valid on the limited class of diagonalizable matrices or on irreducible nonnegative matrices, whose spectral radius has been proved to be a simple eigenvalue.

In this paper, we prove some new formulas bounding the spectral radius of any nonnegative matrix. They are expressed in terms of the elements of the matrix and they extend previous results, [4, 11]. In the remainder, we give the necessary notation required for our results. For $1 \leq i \leq n$, the quantities

$$r_i(A) = \sum_{j=1}^{n} a_{ij} \quad \text{and} \quad M_i(A) = \sum_{j=1}^{n} a_{ij} r_j(A)$$

are known as the i-th row and the i-th 2-row sums of A, respectively, and for $r_i(A) > 0$ their ratio

$$m_i(A) = \frac{M_i(A)}{r_i(A)}$$

is called the i-th average 2-row sum of A, (see [9, 11]). Motivated by the expression of $m_i(A)$, we define a new quantity as the ratio

$$w_i(A) = \frac{\sum_{j=1}^{n} a_{ij} m_j(A)}{m_i(A)}, \tag{1.1}$$

which can be seen as a further development of the quantity $m_i(A)$ and it will be used to compute new bounds for the spectral radius of a nonnegative matrix. Moreover, the following specific entries of matrix A will be used: its largest diagonal and off-diagonal elements,

$$M = \max_{1 \leq i \leq n}\{a_{ii}\} \quad \text{and} \quad N = \max_{\substack{1 \leq i,j \leq n \\ i \neq j}}\{a_{ij}\},$$

respectively, as well as, its smallest diagonal and off-diagonal elements,

$$S = \min_{1 \leq i \leq n}\{a_{ii}\} \quad \text{and} \quad T = \min_{\substack{1 \leq i,j \leq n \\ i \neq j}}\{a_{ij}\},$$

respectively.

In this article we take into account the new quantities $w_i(A)$, $i = 1, \ldots, n$, defined in (1.1), which can be considered as an average of averages 2-row, to present some new bounds for the spectral radius of a nonnegative matrix. Analytically, in section 2 we prove sharp upper bounds of $\rho(A)$ with respect to $w_i(A)$, while in section 3 we turn our attention to a sharp lower bound of $\rho(A)$. In both sections we characterize the equality cases of the bounds if the matrix is irreducible. Illustrative numerical examples are also provided testing our results and comparing with known bounds.

2. Upper bound for the spectral radius of nonnegative matrices

In this section, we investigate some upper bounds for the spectral radius $\rho(A)$ of a nonnegative matrix extending established results. Motivated by [4, 9, 11] and adopting the techniques used therein, we obtain a new expression for a sharp upper bound of $\rho(A)$. In addition, we compare our findings with the ones presented in [4, 11] providing illustrative examples.

The next lemmas demonstrate well-established bounds for $\rho(A)$, and since they are used in our arguments, they are stated here for the sake of completeness.

Lemma 1. ([5]) *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with i-th row sum $r_i(A)$, $i = 1, \ldots, n$ and largest diagonal element M. Then*

$$\rho(A) \geq M,$$

and

$$\min_{1\leq i\leq n} \{r_i(A)\} \leq \rho(A) \leq \max_{1\leq i\leq n} \{r_i(A)\}.$$

Moreover, if A is irreducible, then either equality holds if and only if $r_1(A) = \cdots = r_n(A)$.

Lemma 2. ([8, Theorem 8.1.26, Corollary 8.1.31]) *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$. Then for any positive vector $x \in \mathbb{R}^n$ we have*

$$\min_{1\leq i\leq n} \left\{ \frac{1}{x_i} \sum_{j=1}^{n} a_{ij} x_j \right\} \leq \rho(A) \leq \max_{1\leq i\leq n} \left\{ \frac{1}{x_i} \sum_{j=1}^{n} a_{ij} x_j \right\}.$$

If A is also irreducible, then

$$\rho(A) = \max_{x>0} \min_{1\leq i\leq n} \left\{ \frac{1}{x_i} \sum_{j=1}^{n} a_{ij} x_j \right\} = \min_{x>0} \max_{1\leq i\leq n} \left\{ \frac{1}{x_i} \sum_{j=1}^{n} a_{ij} x_j \right\}.$$

Combining the definition of $w_i(A)$ in (1.1) with Lemma 2 by taking a vector x with components $m_i(A) > 0$, $1 \leq i \leq n$, we immediately obtain the following proposition, which locates the spectral radius of a nonnegative matrix among the maximum and minimum values of $w_i(A)$.

Proposition 3. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$. Then*

$$\min_{1\leq i\leq n} \{w_i(A)\} \leq \rho(A) \leq \max_{1\leq i\leq n} \{w_i(A)\}.$$

If A is also irreducible, then either equality holds if and only if $w_1(A) = \cdots = w_n(A)$.

Next we state and prove the main result in this section.

Theorem 4. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with row sums $r_i(A) > 0$ and average 2-row sums $m_i(A) > 0$ for all $i = 1, \ldots, n$, and $w_i(A)$ as defined in (1.1) such that $w_1(A) \geq w_2(A) \geq \cdots \geq w_n(A) > 0$. Let M, N be the largest diagonal and off-diagonal elements of A, respectively, with $N > 0$. Denoting by $b_m = \max\left\{\frac{m_j(A)}{m_i(A)} : 1 \leq i, j \leq n, \ i \neq j\right\}$, consider*

$$\Psi_\ell = \frac{1}{2}\left(w_\ell(A) + M - Nb_m + \sqrt{\Delta_\ell}\right), \ \ell = 1, \ldots, n, \tag{2.1}$$

where

$$\Delta_\ell = (w_\ell(A) - M + Nb_m)^2 + 4Nb_m \sum_{j=1}^{\ell-1}(w_j(A) - w_\ell(A)). \tag{2.2}$$

Then

$$\rho(A) \le \min\{\Psi_\ell : \quad 1 \le \ell \le n\}. \tag{2.3}$$

Proof. To simplify the exposition of our calculations, we let $m_i = m_i(A)$, and $w_i = w_i(A)$ for $1 \le i \le n$.

Consider $\ell = 1$. Due to Lemma 1, Proposition 3 and our assumptions,

$$M \le \rho(A) \le \max_{1\le i\le n}\{w_i\} \equiv w_1,$$

which yield that

$$w_1 - M \ge 0 \Rightarrow w_1 - M + Nb_m \ge Nb_m > 0.$$

Then the quantities in (2.1), (2.2) give

$$\begin{aligned}\Psi_1 &= \frac{1}{2}\left(w_1 + M - Nb_m + \sqrt{(w_1 - M + Nb_m)^2}\right)\\ &= \frac{1}{2}(w_1 + M - Nb_m + w_1 - M + Nb_m)\\ &= w_1.\end{aligned}$$

Then $\rho(A) \le \Psi_1$.

Consider $2 \le \ell \le n$. Let $U = \mathrm{diag}(m_1x_1, \ldots, m_{\ell-1}x_{\ell-1}, m_\ell, \ldots, m_n)$ be an $n \times n$ diagonal matrix, where $x_j \ge 1$ is a variable to be determined later for $1 \le j \le \ell - 1$ and let $B = U^{-1}AU$. Due to similarity, A and B have the same eigenvalues, hence $\rho(A) = \rho(B)$.

For $1 \le i \le \ell - 1$, we derive

$$\begin{aligned}r_i(B) &= r_i(U^{-1}AU) = \frac{1}{x_i}\left(\sum_{j=1}^{\ell-1} a_{ij}\frac{m_j}{m_i}x_j + \sum_{j=\ell}^{n} a_{ij}\frac{m_j}{m_i}\right)\\ &= \frac{1}{x_i}\left(\sum_{\substack{j=1\\ j\ne i}}^{\ell-1} a_{ij}\frac{m_j}{m_i}x_j + a_{ii}x_i + \sum_{j=1}^{n} a_{ij}\frac{m_j}{m_i} - \sum_{j=1}^{\ell-1} a_{ij}\frac{m_j}{m_i}\right)\\ &= \frac{1}{x_i}\left(\sum_{j=1}^{n} a_{ij}\frac{m_j}{m_i} + \sum_{\substack{j=1\\ j\ne i}}^{\ell-1} a_{ij}\frac{m_j}{m_i}x_j - \sum_{\substack{j=1\\ j\ne i}}^{\ell-1} a_{ij}\frac{m_j}{m_i} + a_{ii}x_i - a_{ii}\right)\\ &= \frac{1}{x_i}\left(\sum_{j=1}^{n} a_{ij}\frac{m_j}{m_i} + \sum_{\substack{j=1\\ j\ne i}}^{\ell-1} a_{ij}\frac{m_j}{m_i}(x_j - 1) + a_{ii}(x_i - 1)\right).\end{aligned} \tag{2.4}$$

Obviously, $a_{ii} \le M$, $a_{ij} \le N$, and $\frac{m_j}{m_i} \le b_m$ for $1 \le j \le \ell - 1$ and $j \ne i$. Combining these inequalities with the definition of w_i in (1.1), the equality (2.4) is formulated as

$$r_i(B) \le \frac{1}{x_i}\left(w_i + Nb_m \sum_{\substack{j=1\\ j\ne i}}^{\ell-1}(x_j - 1) + M(x_i - 1)\right), \; i = 1, \ldots, \ell - 1. \tag{2.5}$$

For $\ell \le i \le n$, we derive

$$\begin{aligned} r_i(B) &= r_i(U^{-1}AU) = \sum_{j=1}^{\ell-1} a_{ij}\frac{m_j}{m_i}x_j + \sum_{j=\ell}^{n} a_{ij}\frac{m_j}{m_i} \\ &= \sum_{j=1}^{\ell-1} a_{ij}\frac{m_j}{m_i}x_j + \sum_{j=1}^{n} a_{ij}\frac{m_j}{m_i} - \sum_{j=1}^{\ell-1} a_{ij}\frac{m_j}{m_i} \\ &= \sum_{j=1}^{n} a_{ij}\frac{m_j}{m_i} + \sum_{j=1}^{\ell-1} a_{ij}\frac{m_j}{m_i}(x_j - 1) \qquad (2.6) \\ &\le w_\ell + Nb_m \sum_{j=1}^{\ell-1}(x_j - 1). \qquad (2.7) \end{aligned}$$

At this point, in order to construct the variable x_j for $j = 1, \ldots, \ell - 1$ and $\ell = 2, \ldots, n$, we consider the real roots of the quadratic equations

$$\Psi_\ell^2 - (w_\ell + M - Nb_m)\Psi_\ell + w_\ell(M - Nb_m) - Nb_m \sum_{j=1}^{\ell-1}(w_j - w_\ell) = 0. \qquad (2.8)$$

In particular, the trinomials in (2.8) have discriminant

$$\begin{aligned} \Delta_\ell &\equiv (w_\ell + M - Nb_m)^2 - 4\left(w_\ell(M - Nb_m) - Nb_m \sum_{j=1}^{\ell-1}(w_j - w_\ell)\right) \\ &= (w_\ell - M + Nb_m)^2 + 4Nb_m \sum_{j=1}^{\ell-1}(w_j - w_\ell). \end{aligned}$$

Due to the hypotheses that $N > 0$, $b_m > 0$ and $\{w_i\}_{i=1}^n$ is a decreasing sequence of positive numbers, the discriminant Δ_ℓ is a positive number for all $\ell = 2, \ldots, n$, which yields that the quadratic equations in (2.8) have two distinct real roots with a positive one

$$\Psi_\ell = \frac{1}{2}\left(w_\ell + M - Nb_m + \sqrt{\Delta_\ell}\right), \ \ \ell = 2, \ldots, n. \qquad (2.9)$$

Now, for $1 \le j \le \ell - 1$, we consider

$$x_j = 1 + \frac{w_j - w_\ell}{\Psi_\ell - M + Nb_m} \Leftrightarrow x_j - 1 = \frac{w_j - w_\ell}{\Psi_\ell - M + Nb_m}, \qquad (2.10)$$

where Ψ_ℓ is given by (2.9). If $\sum_{j=1}^{\ell-1}(w_j - w_\ell) > 0$, it is clear by relation (2.9) that

$$\begin{aligned} \Psi_\ell &> \frac{1}{2}\left(w_\ell + M - Nb_m + |w_\ell - M + Nb_m|\right) \\ &\ge \frac{1}{2}\left(w_\ell + M - Nb_m - (w_\ell - M + Nb_m)\right) \\ &= M - Nb_m, \end{aligned}$$

otherwise, $w_1 = \cdots = w_\ell \geq M \Rightarrow w_\ell - M + Nb_m > 0$ and (2.9) yields

$$\begin{aligned}
\Psi_\ell &= \frac{1}{2}\left(w_\ell + M - Nb_m + |w_\ell - M + Nb_m|\right) \\
&> \frac{1}{2}\left(w_\ell + M - Nb_m - (w_\ell - M + Nb_m)\right) \\
&= M - Nb_m.
\end{aligned}$$

Both cases ensure $x_j - 1 \geq 0$ in (2.10).

Additionally, by the expression (2.8), we may write

$$\begin{aligned}
Nb_m \sum_{j=1}^{\ell-1}(w_j - w_\ell) &= \Psi_\ell^2 - (w_\ell + M - Nb_m)\Psi_\ell + w_\ell(M - Nb_m) \\
&= \Psi_\ell(\Psi_\ell - w_\ell - M + Nb_m) + w_\ell(M - Nb_m) \\
&= \Psi_\ell(\Psi_\ell - M + Nb_m) - \Psi_\ell w_\ell + w_\ell(M - Nb_m) \\
&= \Psi_\ell(\Psi_\ell - M + Nb_m) - w_\ell(\Psi_\ell - M + Nb_m) \\
&= (\Psi_\ell - M + Nb_m)(\Psi_\ell - w_\ell). \qquad (2.11)
\end{aligned}$$

Overall, we take $x_j - 1 \geq 0$, $j = 1, \ldots, \ell - 1$ from (2.10) and $Nb_m \sum_{j=1}^{\ell-1}(w_j - w_\ell)$ from (2.11) and substitute them into the inequality (2.5). Hence, for $1 \leq i \leq \ell - 1$, we obtain

$$\begin{aligned}
r_i(B) &\leq \frac{1}{x_i}\left(w_i + Nb_m \sum_{\substack{j=1 \\ j\neq i}}^{\ell-1}(x_j - 1) + M(x_i - 1)\right) \\
&= \frac{1}{x_i}\left(w_i + Nb_m \sum_{j=1}^{\ell-1}(x_j - 1) + M(x_i - 1) - Nb_m(x_i - 1)\right) \\
&= \frac{1}{x_i}\left(w_i + Nb_m \sum_{j=1}^{\ell-1}(x_j - 1) + (M - Nb_m)(x_i - 1)\right) \\
&= \frac{1}{x_i}\left(w_i + Nb_m \sum_{j=1}^{\ell-1}\frac{w_j - w_\ell}{\Psi_\ell - M + Nb_m} + (M - Nb_m)\frac{w_i - w_\ell}{\Psi_\ell - M + Nb_m}\right) \\
&= \frac{1}{x_i}\left(w_i + \frac{(\Psi_\ell - M + Nb_m)(\Psi_\ell - w_\ell)}{\Psi_\ell - M + Nb_m} + (M - Nb_m)\frac{w_i - w_\ell}{\Psi_\ell - M + Nb_m}\right) \\
&= \frac{w_i(\Psi_\ell - M + Nb_m) + (\Psi_\ell - M + Nb_m)(\Psi_\ell - w_\ell) + (M - Nb_m)(w_i - w_\ell)}{x_i(\Psi_\ell - M + Nb_m)} \\
&= \frac{(\Psi_\ell - M + Nb_m)\Psi_\ell + (w_i - w_\ell)(\Psi_\ell - M + Nb_m + M - Nb_m)}{x_i(\Psi_\ell - M + Nb_m)} \\
&= \frac{(\Psi_\ell - M + Nb_m)\Psi_\ell + (w_i - w_\ell)\Psi_\ell}{\frac{\Psi_\ell - M + Nb_m + w_i - w_\ell}{\Psi_\ell - M + Nb_m}(\Psi_\ell - M + Nb_m)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(\Psi_\ell - M + Nb_m + w_i - w_\ell)\Psi_\ell}{\Psi_\ell - M + Nb_m + w_i - w_\ell} \\
&= \Psi_\ell, \qquad (2.12)
\end{aligned}
$$

where $\ell = 2, \ldots, n$.

Similarly, for $\ell \leq i \leq n$ and $1 \leq j \leq \ell - 1$ we substitute the relations (2.10) and (2.11) into the inequality (2.7), which can be written as

$$
\begin{aligned}
r_i(B) &\leq w_i + Nb_m \sum_{j=1}^{\ell-1} (x_j - 1) \\
&= w_i + \frac{Nb_m}{\Psi_\ell - M + Nb_m} \sum_{j=1}^{\ell-1} (w_j - w_\ell) \\
&= w_i + \frac{(\Psi_\ell - M + Nb_m)(\Psi_\ell - w_\ell)}{\Psi_\ell - M + Nb_m} \\
&= w_i + \Psi_\ell - w_\ell \\
&\leq w_\ell + \Psi_\ell - w_\ell = \Psi_\ell. \qquad (2.13)
\end{aligned}
$$

Thus, for all $2 \leq \ell \leq n$ and $1 \leq i \leq n$ the inequalities (2.12) and (2.13) verify

$$
r_i(B) \leq \Psi_\ell,
$$

and by Lemma 1,

$$
\rho(A) = \rho(B) \leq \max_{1 \leq i \leq n} \{r_i(B)\} \leq \Psi_\ell. \qquad (2.14)
$$

The validity of (2.3) follows readily from (2.14). □

Remark 1. Obviously, inequalities (2.5) and (2.7) are stated as equalities for the special values of M, N, b_m, x_i, as the following cases indicate.

(i) For $1 \leq i \leq \ell - 1$, (2.5) is given by equality if and only if (a) and (b) hold:
  - (a) $x_i = 1$ or $a_{ii} = M$, when $x_i > 1$,
  - (b) $x_j = 1$ or $a_{ij} = N$ and $\frac{m_j}{m_i} = b_m$, when $x_j > 1$ with $1 \leq j \leq \ell - 1$ and $j \neq i$.

(ii) For $\ell \leq i \leq n$, (2.7) is given by equality if and only if (c) and (d) hold:
  - (c) $x_j = 1$ or $a_{ij} = N$ and $\frac{m_j}{m_i} = b_m$, when $x_j > 1$ with $1 \leq j \leq \ell - 1$,
  - (d) $w_i = w_\ell$.

Using Remark 1 and Proposition 3 for a nonnegative and irreducible matrix, a special formulation of the spectral radius of the matrix is derived shown in the following proposition.

Proposition 5. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be nonnegative and irreducible and let the quantities M, N, b_m, Ψ_ℓ, $m_i(A), w_i(A)$, $i = 1, \ldots, n$ satisfy the notations and assumptions of Theorem 4. Then $\rho(A) = \Psi_\ell$ holds for some $\ell = 1, \ldots, n$ if and only if $w_1(A) = \cdots = w_n(A) > 0$ or for some $t = 2, \ldots, \ell$, A satisfies the following conditions:*

(i) $a_{ii} = M$, *for* $1 \leq i \leq t-1$,
(ii) $a_{ij} = N$ *and* $\frac{m_j(A)}{m_i(A)} = b_m$, *for* $1 \leq i \leq n$, $1 \leq j \leq t-1$ *and* $j \neq i$,
(iii) $w_t(A) = \cdots = w_n(A)$.

Proof. Suppose that A is nonnegative and irreducible with $\rho(A) = \Psi_\ell$ for some $2 \leq \ell \leq n$. Consider $B = U^{-1}AU$ as constructed in the proof of Theorem 4, then B is also nonnegative and irreducible with $\rho(B) = \max_{1\leq i\leq n} r_i(B) = \rho(A) = \Psi_\ell$. By Lemma 1, $r_1(B) = \cdots = r_n(B) = \Psi_\ell$, and thus, for $1 \leq i \leq \ell - 1$ cases (a) and (b) and for $\ell \leq i \leq n$ cases (c) and (d) in Remark 1 hold. Let us firstly assume that $w_1(A) = w_\ell(A)$, then case (d) implies $w_1(A) = \cdots = w_n(A)$. On the other hand, if $w_1(A) > w_\ell(A)$, consider the smallest integer $2 \leq t \leq \ell$ such that $w_t(A) = w_\ell(A)$. Clearly, $w_i(A) > w_\ell(A)$ for integers $1 \leq i \leq t-1$, which implies that $x_i > 1$. Therefore, conditions (i) and (ii) follow from the corresponding cases (a) and (b) of Remark 1 for $1 \leq i \leq \ell - 1$ and the case (c) for $\ell \leq i \leq n$. Condition (iii) is verified by case (d), since we have $w_t(A) = \cdots = w_\ell(A) = \cdots = w_n(A)$.
For the converse statement, if $w_1(A) = \cdots = w_n(A)$, then for $1 \leq \ell \leq n$ we obtain $\Psi_\ell = w_1(A)$ and by Proposition 3, $\rho(A) = w_1(A) = \Psi_\ell$. If conditions (i)–(iii) hold for some $2 \leq t \leq \ell \leq n$, then (a) and (b) hold for $1 \leq i \leq \ell - 1$, and (c) and (d) hold for $\ell \leq i \leq n$, implying that $r_i(B) = \Psi_\ell$ for $1 \leq i \leq n$, and thus by Lemma 1, $\rho(A) = \rho(B) = \Psi_\ell$. □

Next we present a sharp upper bound for the spectral radius of a nonnegative matrix proved by Duan, Zhou in [4] and Xing, Zhou in [11]. We outline the corresponding theorems, since their expressions will be compared to our results proved in Theorem 4.

Theorem 6. ([4, Theorem 2.1]) *Let* $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ *with row sums* $r_1(A) \geq r_2(A) \geq \cdots \geq r_n(A)$. *Let* M, N *be the largest diagonal and non-diagonal elements of* A, *respectively. Suppose that* $N > 0$. *For* $1 \leq \ell \leq n$, *let*

$$\hat{\Phi}_\ell = \frac{r_\ell(A) + M - N}{2} + \sqrt{\left(\frac{r_\ell(A) - M + N}{2}\right)^2 + N\sum_{i=1}^{\ell-1}(r_i(A) - r_\ell(A))}. \tag{2.15}$$

Then, $\rho(A) \leq \hat{\Phi}_\ell$ *for* $1 \leq \ell \leq n$.

Theorem 7. ([11, Theorem 2.1]) *Let* $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ *with row sums* $r_i(A) > 0$, $i = 1, \ldots, n$ *and average 2-row sums* $m_1(A) \geq m_2(A) \geq \cdots \geq m_n(A)$. *Let* M, N *be the largest diagonal and non-diagonal elements of* A, *respectively. Suppose that* $N > 0$. *Let* $b = \max\left\{\frac{r_j(A)}{r_i(A)} : \ 1 \leq i, j \leq n\right\}$ *and*

$$\widetilde{\Phi}_\ell = \frac{m_\ell(A) + M - Nb}{2} + \sqrt{\left(\frac{m_\ell(A) - M + Nb}{2}\right)^2 + Nb\sum_{i=1}^{\ell-1}(m_i(A) - m_\ell(A))},$$

for $1 \leq \ell \leq n$. *Then,* $\rho(A) \leq \widetilde{\Phi}_\ell$ *for* $1 \leq \ell \leq n$.

At the following example we test the results of Theorem 4 in comparison to the ones of Theorems 6 and 7.

Example 8. Consider the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The spectrum of A is $\sigma(A) = \{-0.8951, 1, 2.6027, 4.2924\}$, which means that $\rho(A) = 4.2924$. Clearly, its row sums are $r_1(A) = r_2(A) = 5, r_3(A) = 4, r_4(A) = 1$, its average 2-row sums are $m_1(A) = 4.8, m_2(A) = 4, m_3(A) = 5, m_4(A) = 1$ and $w_1(A) = 4.875, w_2(A) = 4.5, w_3(A) = 3.52, w_4(A) = 1$, $M = 3$, $N = 2$, and $b_m = 5$. The assumptions of Theorem 4 hold and the quantities Ψ_ℓ, $1 \le \ell \le 4$, given by (2.1), are $\Psi_1 = 4.875, \Psi_2 = 4.8173, \Psi_3 = 5.4027$, and $\Psi_4 = 7.7215$. Hence, the inequality (2.3) yields $\rho(A) \le 4.8173$.

The assumptions of Theorem 6 are also satisfied, since $r_1(A) = r_2(A) > r_3(A) > r_4(A)$ and the quantities $\hat{\Phi}_\ell$, given by (2.15), are $\hat{\Phi}_1 = \hat{\Phi}_2 = \hat{\Phi}_3 = 5$, and $\hat{\Phi}_4 = 5.6904$. Then, Theorem 6 yields $\rho(A) \le 5$.

In order to apply Theorem 7, we use the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

such that A is similar to

$$B = PAP^T = \begin{pmatrix} 0 & 2 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the assumptions of Theorem 7 hold for the matrix B, since $r_1(B) = 4$, $r_2(B) = r_3(B) = 5, r_4(B) = 1$, $m_1(B) = 5, m_2(B) = 4.8$, $m_3(B) = 4, m_4(B) = 1$ and $M = 3$, $N = 2$, $b = 5$; the quantities $\widetilde{\Phi}_\ell$ are computed to be equal to $\widetilde{\Phi}_1 = 5, \widetilde{\Phi}_2 = 4.9671, \widetilde{\Phi}_3 = 5.4462$, and $\widetilde{\Phi}_4 = 8.1355$. Due to the similarity of matrices A, B and Theorem 7, $\rho(A) = \rho(B) \le 4.9671$.

Overall, Theorem 4 appears to be a refinement, since the upper bound of the spectral radius $\rho(A)$ computed by the expressions of Theorem 4 is sharper than the ones computed by Theorems 6 and 7.

At this point, let us consider some well-known nonnegative matrices representing connectivity properties of finite graphs. Let G be a simple undirected graph on vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$ with no isolated vertices. Recall that the adjacency matrix $A(G) \in \mathcal{M}_n(\mathbb{R})$ of G is a symmetric $(0, 1)$-matrix, whose entries depend on whether the corresponding vertices are adjacent and the degree matrix $D(G) = diag(d_1, \ldots, d_n) \in \mathcal{M}_n(\mathbb{R})$ is the diagonal matrix with vertex degrees of G. Both of these matrices are linked to the signless Laplacian matrix of the graph defined as $Q(G) = D(G) + A(G)$. In particular, the entries q_{ij} of $Q(G)$ are given by

$$q_{ij} = \begin{cases} d_i, & \text{if } i = j, \\ 1, & \text{if } i \neq j \text{ and } v_i v_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Apparently, $Q(G)$ is a symmetric positive semi-definite matrix whose spectral radius $\rho(Q(G))$ is known as the signless Laplacian spectral radius of G. More information about the graph can be provided by the average 2-degree of vertex $v_i \in V(G)$, $m_i = d_i^{-1} \sum_{j:v_i v_j \in E(G)} d_j$. The sequence $\{(d_i, m_i)\}_{i=1}^n$ of pairs is called the sequence of degree pairs of G.

The next example illustrates the formulas proved in Theorem 4 applied on the signless Laplacian matrix of an undirected graph as defined above.

Example 9. Let an undirected graph G with signless Laplacian matrix

$$Q(G) = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

The spectrum of $Q(G)$ is $\sigma(Q(G)) = \{0.7639, 2, 5.2361\}$, which means that $\rho(Q(G)) = 5.2361$. Then $r_1(Q(G)) = r_2(Q(G)) = 6$, $r_3(Q(G)) = r_4(Q(G)) = 4$, $m_1(Q(G)) = m_2(Q(G)) = 5.3333$, $m_3(Q(G)) = m_4(Q(G)) = 5$ and $w_1(Q(G)) = w_2(Q(G)) = 5.8750$, $w_3(Q(G)) = w_4(Q(G)) = 4.1333$, $M = 3$, $N = 1$, and $b_m = 1.0667$. Apparently, the assumptions of Theorem 4 are satisfied and the quantities Ψ_ℓ, $1 \leq \ell \leq 4$, given by (2.1), are $\Psi_1 = \Psi_2 = 5.8750$ and $\Psi_3 = \Psi_4 = 5.2527$. Hence, the inequality (2.3) yields $\rho(Q(G)) \leq 5.2527$.

The assumptions of Theorem 6 are also satisfied, since $r_1(Q(G)) = r_2(Q(G)) > r_3(Q(G)) = r_4(Q(G))$, and the quantities $\hat{\Phi}_\ell$, given by (2.15), are $\hat{\Phi}_1 = \hat{\Phi}_2 = 6$, $\hat{\Phi}_3 = \hat{\Phi}_4 = 5.2361$. Thus, Theorem 6 yields $\rho(A_2) \leq 5.2361$.

The assumptions of Theorem 7 also hold, since $m_1(Q(G)) = m_2(Q(G)) > m_3(Q(G)) = m_4(Q(G))$, and the quantities $\widetilde{\Phi}_\ell$ are equal to $\widetilde{\Phi}_1 = \widetilde{\Phi}_2 = 5.3333$ and $\widetilde{\Phi}_3 = \widetilde{\Phi}_4 = 5.2656$. Evidently, $\rho(Q(G)) \leq 5.2656$.

Concluding, we notice that the upper bound of $\rho(Q(G))$ calculated by Theorem 6 coincides with the exact value of $\rho(Q(G))$, which reveals that in this case Theorem 6 provides a sharper estimate than the corresponding ones computed by Theorems 4 and 7.

3. Lower bound for the spectral radius of nonnegative matrices

In this section, we obtain a new result on the lower bound of the spectral radius of nonnegative matrices, and we compare these with the bounds investigated in [4, 11].

Theorem 10. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with row sums $r_i(A) > 0$ and average 2-row sums $m_i(A) > 0$ for all $i = 1, \ldots, n$, and $w_i(A)$ as defined in (1.1) such that $w_1(A) \geq w_2(A) \geq \cdots \geq w_n(A) > 0$. Let S, T be the smallest diagonal and off-diagonal elements of A, respectively. Denoting by $c_m = \min\left\{\frac{m_j(A)}{m_i(A)} : \ 1 \leq i, j \leq n, \ \ i \neq j\right\}$, consider*

$$\psi_n = \frac{1}{2}\left(w_n(A) + S - Tc_m + \sqrt{\Delta_n}\right), \tag{3.1}$$

where

$$\Delta_n = (w_n(A) - S + Tc_m)^2 + 4Tc_m \sum_{j=1}^{n-1}(w_j(A) - w_n(A)). \tag{3.2}$$

Then

$$\rho(A) \geq \psi_n. \tag{3.3}$$

Proof. To simplify the exposition of our calculations, we let $m_i = m_i(A)$, and $w_i = w_i(A)$ for $1 \leq i \leq n$.

If $T = 0$, equality (3.1) degenerates to $\psi_n = w_n$ due to the fact that $w_n \geq a_{nn} \geq S$ and Theorem 10 is apparent from Proposition 3. Consequently, in what follows we assume $T > 0$.

Let $U = \mathrm{diag}(m_1x_1, m_2x_2, \ldots, m_{n-1}x_{n-1}, m_n)$ be an $n \times n$ diagonal matrix, where $x_j \geq 1$ for $1 \leq j \leq n-1$ is a variable to be determined later and let $B = U^{-1}AU$. Due to similarity, A and B have the same eigenvalues, hence, $\rho(A) = \rho(B)$.

For $1 \leq i \leq n-1$ we refer to the equality (2.4) with $\ell = n$, that is,

$$r_i(B) = r_i(U^{-1}AU) = \frac{1}{x_i}\left(\sum_{j=1}^{n} a_{ij}\frac{m_j}{m_i} + \sum_{\substack{j=1\\ j\neq i}}^{n-1} a_{ij}\frac{m_j}{m_i}(x_j-1) + a_{ii}(x_i-1)\right).$$

Moreover, $a_{ii} \geq S$, $a_{ij} \geq T$, and $\frac{m_j}{m_i} \geq c_m$, for $1 \leq j \leq n-1$ and $j \neq i$. Combining these inequalities with the definition of w_i in (1.1), the latter equality is formulated as

$$r_i(B) \geq \frac{1}{x_i}\left(w_i + Tc_m\sum_{\substack{j=1\\ j\neq i}}^{n-1}(x_j-1) + S(x_i-1)\right). \tag{3.4}$$

Furthermore, for the special case $i = n$ the equality (2.6) can be written as

$$\begin{aligned} r_n(B) &= r_n(U^{-1}AU) = \sum_{j=1}^{n-1} a_{nj}\frac{m_j}{m_n}(x_j-1) + \sum_{j=1}^{n} a_{nj}\frac{m_j}{m_n} \\ &= \sum_{j=1}^{n-1} a_{nj}\frac{m_j}{m_n}(x_j-1) + w_n \\ &\geq Tc_m\sum_{j=1}^{n-1}(x_j-1) + w_n. \end{aligned} \tag{3.5}$$

Now, the variable x_j will be formed by the roots of the equation

$$\psi_n^2 - (w_n + S - Tc_m)\psi_n + w_n(S - Tc_m) - Tc_m\sum_{j=1}^{n-1}(w_j - w_n) = 0. \tag{3.6}$$

The quadratic equation in (3.6) has real roots, since its discriminant

$$\begin{aligned} \Delta_n &\equiv (w_n + S - Tc_m)^2 - 4\left(w_n(S - Tc_m) - Tc_m\sum_{j=1}^{n-1}(w_j - w_n)\right) \\ &= (w_n - S + Tc_m)^2 + 4Tc_m\sum_{j=1}^{n-1}(w_j - w_n) \end{aligned}$$

is a positive number, due to $T > 0$, $c_m > 0$ and the monotonicity of the sequence $\{w_i\}_{i=1}^{n}$ of positive numbers. Hence, (3.6) has a positive real root

$$\psi_n = \frac{1}{2}\left(w_n + S - Tc_m + \sqrt{\Delta_n}\right), \tag{3.7}$$

which is used in the construction of

$$x_j = 1 + \frac{w_j - w_n}{\psi_n - S + Tc_m} \Leftrightarrow x_j - 1 = \frac{w_j - w_n}{\psi_n - S + Tc_m}, \tag{3.8}$$

for $1 \le j \le n-1$. If $\sum_{j=1}^{n-1}(w_j - w_n) > 0$, it is clear by relation (3.7) that

$$\begin{aligned} \psi_n &> \frac{1}{2}(w_n + S - Tc_m + |w_n - S + Tc_m|) \\ &\ge \frac{1}{2}(w_n + S - Tc_m - (w_n - S + Tc_m)) \\ &= S - Tc_m, \end{aligned}$$

otherwise, $w_1 = \cdots = w_n \ge S \Rightarrow w_n - S + Tc_m > 0$ and (3.7) yields

$$\begin{aligned} \psi_n &= \frac{1}{2}(w_n + S - Tc_m + |w_n - S + Tc_m|) \\ &> \frac{1}{2}(w_n + S - Tc_m - (w_n - S + Tc_m)) \\ &= S - Tc_m. \end{aligned}$$

Both cases ensure $x_j - 1 \ge 0$ in (3.8).

Moreover, from (3.6) we derive

$$\begin{aligned} Tc_m \sum_{j=1}^{n-1}(w_j - w_n) &= \psi_n^2 - (w_n + S - Tc_m)\psi_n + w_n(S - Tc_m) \\ &= \psi_n(\psi_n - w_n - S + Tc_m) + w_n(S - Tc_m) \\ &= \psi_n(\psi_n - S + Tc_m) - \psi_n w_n + w_n(S - Tc_m) \\ &= \psi_n(\psi_n - S + Tc_m) - w_n(\psi_n - S + Tc_m) \\ &= (\psi_n - S + Tc_m)(\psi_n - w_n). \end{aligned} \tag{3.9}$$

Overall, we substitute $x_j - 1 \ge 0$, $j = 1, \ldots, n-1$, from (3.8) and $Tc_m \sum_{j=1}^{n-1}(w_j - w_n)$ from (3.9) into the inequality (3.4). Hence, for $1 \le i \le n-1$, we have

$$\begin{aligned} r_i(B) &\ge \frac{1}{x_i}\left(w_i + Tc_m \sum_{\substack{j=1 \\ j \ne i}}^{n-1}(x_j - 1) + S(x_i - 1) \right) \\ &= \frac{1}{x_i}\left(w_i + Tc_m \sum_{j=1}^{n-1}(x_j - 1) + S(x_i - 1) - Tc_m(x_i - 1) \right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{x_i}\left(w_i + Tc_m\sum_{j=1}^{n-1}(x_j-1) + (S-Tc_m)(x_i-1)\right)\\
&= \frac{1}{x_i}\left(w_i + Tc_m\sum_{j=1}^{n-1}\frac{w_j-w_n}{\psi_n-S+Tc_m} + (S-Tc_m)\frac{w_i-w_n}{\psi_n-S+Tc_m}\right)\\
&= \frac{1}{x_i}\left(w_i + \frac{(\psi_n-S+Tc_m)(\psi_n-w_n)}{\psi_n-S+Tc_m} + \frac{(S-Tc_m)(w_i-w_n)}{\psi_n-S+Tc_m}\right)\\
&= \frac{w_i(\psi_n-S+Tc_m)+(\psi_n-S+Tc_m)(\psi_n-w_n)+(S-Tc_m)(w_i-w_n)}{x_i(\psi_n-S+Tc_m)}\\
&= \frac{(\psi_n-S+Tc_m)(\psi_n+w_i-w_n)+(S-Tc_m)(w_i-w_n)}{x_i(\psi_n-S+Tc_m)}\\
&= \frac{\psi_n(\psi_n-S+Tc_m)+\psi_n(w_i-w_n)}{x_i(\psi_n-S+Tc_m)}\\
&= \frac{\psi_n(\psi_n-S+Tc_m+w_i-w_n)}{x_i(\psi_n-S+Tc_m)}\\
&= \frac{\psi_n(\psi_n-S+Tc_m+w_i-w_n)}{\frac{\psi_n-S+Tc_m+w_i-w_n}{\psi_n-S+Tc_m}(\psi_n-S+Tc_m)}\\
&= \psi_n. \qquad (3.10)
\end{aligned}
$$

Also, by substituting $x_j - 1 \geq 0$ and $Tc_m \sum_{j=1}^{n-1}(w_j - w_n)$ from (3.8) and (3.9), respectively, into the inequality (3.5), we may write

$$
\begin{aligned}
r_n(B) &\geq w_n + Tc_m\sum_{j=1}^{n-1}\frac{w_j-w_n}{\psi_n-S+Tc_m}\\
&= w_n + \frac{Tc_m}{\psi_n-S+Tc_m}\sum_{j=1}^{n-1}(w_j-w_n)\\
&= w_n + \frac{(\psi_n-S+Tc_m)(\psi_n-w_n)}{\psi_n-S+Tc_m}\\
&= \psi_n. \qquad (3.11)
\end{aligned}
$$

Hence, for all $1 \leq i \leq n$ the inequalities (3.10) and (3.11) confirm

$$r_i(B) \geq \psi_n.$$

By Lemma 1,

$$\rho(A) = \rho(B) \geq \min_{1\leq i\leq n}\{r_i(B)\} \geq \psi_n,$$

verifying the validity of (3.3). □

Analogously to equality cases stated in Remark 1 for the upper bounds of the spectral radius of a nonnegative matrix, we may have corresponding equality cases for the lower bounds as stated in the following arguments.

Remark 2. Inequalities (3.4) and (3.5) are reduced to equalities for the special values of S, T, c_m, x_i, as the following cases indicate.

(i) For $1 \leq i \leq n-1$, (3.4) is given by equality if and only if (a) and (b) hold:

(a) $x_i = 1$ or $a_{ii} = S$, when $x_i > 1$,

(b) $x_j = 1$ or $a_{ij} = T$ and $\frac{m_j}{m_i} = c_m$, when $x_j > 1$ with $1 \leq j \leq n-1$ and $j \neq i$.

(ii) Inequality (3.5) is given by equality if and only if (c) holds:

(c) $x_j = 1$ or $a_{nj} = T$ and $\frac{m_j}{m_n} = c_m$, when $x_j > 1$ with $1 \leq j \leq n-1$.

Using Remark 2 and Proposition 3 for a nonnegative and irreducible matrix, a special formulation of the spectral radius of the matrix is derived shown in the following proposition.

Proposition 11. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be nonnegative and irreducible and let the quantities S, T, c_m, ψ_n, $m_i(A), w_i(A)$, $i = 1, \ldots, n$ satisfy the notations and assumptions of Theorem 10. Then $\rho(A) = \psi_n$ holds if and only if $w_1(A) = \cdots = w_n(A) > 0$ or $T > 0$ and for some $t = 2, \ldots, n$, A satisfies the following conditions:*

(i) $a_{ii} = S$, for $1 \leq i \leq t-1$,

(ii) $a_{ij} = T$ and $\frac{m_j(A)}{m_i(A)} = c_m$, for $1 \leq i \leq n$, $1 \leq j \leq t-1$ and $j \neq i$,

(iii) $w_t(A) = \cdots = w_n(A)$,

Proof. Suppose that A is nonnegative and irreducible with $\rho(A) = \psi_n$. Consider $B = U^{-1}AU$ as constructed in the proof of Theorem 10, then B is also nonnegative and irreducible with $\rho(B) = \max_{1 \leq i \leq n} r_i(B) = \rho(A) = \Psi_\ell$. By Lemma 1, $r_1(B) = \cdots = r_n(B) = \psi_n$, and thus, for $1 \leq i \leq n-1$ cases (a) and (b) and for $i = n$ case (c) in Remark 2 hold. If $w_1(A) > w_n(A)$, consider the smallest integer $2 \leq t \leq n$ such that $w_t(A) = w_n(A)$. Clearly, $w_i(A) > w_n(A)$ for $1 \leq i \leq t-1$, which implies that $x_i > 1$. Therefore, conditions (i)-(iii) of Theorem 10 follow from cases (a) and (b) for $1 \leq i \leq n-1$ and (c) of Remark 2.
Conversely, if $w_1(A) = \cdots = w_n(A)$, then by Proposition 3, $\rho(A) = w_n(A) = \psi_n$. If conditions (i)–(iii) hold for some $2 \leq t \leq n$, then (a) and (b) for $1 \leq i \leq n-1$, and (c) hold, implying that $r_i(B) = \psi_n$ for $1 \leq i \leq n$, and thus by Lemma 1, $\rho(A) = \rho(B) = \psi_n$. □

The following statements concern lower bounds for the spectral radius of nonnegative matrices proved by Duan, Zhou [4] and Xing, Zhou [11]. They are presented here for reasons of comparison.

Theorem 12. ([4, Theorem 2.2]) *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with row sums $r_1(A) \geq r_2(A) \geq \cdots \geq r_n(A)$. Let S, T be the smallest diagonal and non-diagonal elements of A, respectively. Let*

$$\hat{\phi}_n = \frac{r_n(A) + S - T}{2} + \sqrt{\left(\frac{r_n(A) - S + T}{2}\right)^2 + T \sum_{i=1}^{n-1} (r_i(A) - r_n(A))}. \tag{3.12}$$

Then, $\rho(A) \geq \hat{\phi}_n$.

Theorem 13. ([11, Theorem 2.3]) *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with all row sums positive and average 2-row sums such that $m_1(A) \geq m_2(A) \geq \cdots \geq m_n(A)$. Let S, T be the smallest diagonal and non-diagonal* *elements of A, respectively. Denote by* $c = \min\left\{\frac{r_j(A)}{r_i(A)} : 1 \le i, j \le n\right\}$ *and by*

$$\widetilde{\phi}_n = \frac{m_n(A) + S - Tc}{2} + \sqrt{\left(\frac{m_n(A) - S + Tc}{2}\right)^2 + Tc\sum_{i=1}^{n-1}(m_i(A) - m_n(A))}. \tag{3.13}$$

Then, $\rho(A) \ge \widetilde{\phi}_n$.

We provide the following illustrative example, comparing the results among Theorems 10, 12 and 13.

Example 14. I. Consider the matrix

$$A = \begin{pmatrix} 6 & 2 & 2 & 2 \\ 0 & 2 & 2 & 1 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{pmatrix}.$$

The spectrum of A is $\sigma(A) = \{-2, -0.7321, 2.7321, 8\}$, which means that $\rho(A) = 8$. With respect to the notations of Theorem 10, $r_1(A) = 12, r_2(A) = 5, r_3(A) = r_4(A) = 6$, $m_1(A) = 8.8333, m_2(A) = 5.6, m_3(A) = m_4(A) = 7.6667$ and $w_1(A) = 10.7396, w_2(A) = 6.1071, w_3(A) = w_4(A) = 5.7652$, and $c_m = 0.6340$. The equality $S = T = 0$ in (3.1) and (3.2) yields $\rho(A) \ge w_4(A) = 5.7652$.

Using the permutation matrix $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, we obtain the similar matrix

$$B = PAP^T = \begin{pmatrix} 6 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix},$$

for which $r_1(B) = 12, r_2(B) = r_3(B) = 6, r_4(B) = 5$, and $m_1(B) = 8.8333, m_2(B) = m_3(B) = 7.6667, m_4(B) = 5.6$, and $S = T = 0$.

Clearly, the assumptions of Theorem 12 are satisfied and by (3.12), $\hat{\phi}_4 = r_4(B) = 5$. Then, the similarity of A, B implies $\rho(A) = \rho(B) \ge 5$.

The assumptions of Theorem 13 are also satisfied and by (3.13), $\widetilde{\phi}_4 = m_4(B) = 5.6$. Then $\rho(A) = \rho(B) \ge 5.6$.

Overall, the lower bound of Theorem 10 appears to be sharper than the ones evaluated by Theorems 12 and 13.

II. Let again the signless Laplacian matrix $Q(G)$ be presented at Example 9 with $\rho(Q(G)) = 5.2361$. Recall the orderings

$$\begin{aligned} r_1(Q(G)) = r_2(Q(G)) &> r_3(Q(G)) = r_4(Q(G)) \\ m_1(Q(G)) = m_2(Q(G)) &> m_3(Q(G)) = m_4(Q(G)) \end{aligned}$$

$$w_1(Q(G)) = w_2(Q(G))) \quad > \quad w_3(Q(G)) = w_4(Q(G)),$$

each of them satisfying the assumption of Theorems 12, 13 and 10, respectively. Morover, $S = 2$, $T = 0$, and $c_m = 0.9375$. Then, Theorem 10 provides the estimate $\rho(Q(G)) \geq \psi_4 = 4.1333$, Theorem 12 yields $\rho(Q(G)) \geq \hat{\phi}_4 = 4$ and lastly, Theorem 13 implies $\rho(Q(G)) \geq \widetilde{\phi}_4 = 5$. In this case, the lower bound of Theorem 13 appears to be sharper than the ones from Theorems 10 and 12.

Conflict of interest

The authors declare no conflict of interest.

References

1. M. Adam, Aik. Aretaki, *Sharp bounds for eigenvalues of the generalized k, m-step Fibonacci matrices*, Proceedings of the 3rd International Conference on Numerical Analysis and Scientific Computation with Applications (NASCA18), Kalamata, Greece, (2018). Available from: `http://nasca18.math.uoa.gr/participants-nbsp.html`.
2. A. Brauer, I.C. Gentry, *Bounds for the greatest characteristic root of an irreducible nonnegative matrix*, Linear Algebra its Appl., **8** (1974), 105–107.
3. F. Duan, K. Zhang, *An algorithm of diagonal transformation for Perron root of nonnegative irreducible matrices*, Appl. Math. Comput., **175** (2006), 762–772.
4. X. Duan, B. Zhou, *Sharp bounds on the spectral radius of a nonnegative matrix*, Linear Algebra its Appl., **439** (2013), 2961–2970.
5. G. Frobenius, *Über Matrizen aus nicht negativen Elementen*, Sitzungsber, Kön. Preuss. Akad. Wiss. Berlin, (1912), 465–477.
6. J. He, Y.M. Liu, J.K. Tian, et al., *Some new sharp bounds for the spectral radius of a nonnegative matrix and its application*, J. Inequalities Appl., **260** (2017), 1–6.
7. W. Hong, L. You, *Further results on the spectral radius of matrices and graphs*, Appl. Math. Comput., **239** (2014), 326–332.
8. R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, second edition, 2013.
9. H. Lin, B. Zhou, *On sharp bounds for spectral radius of nonnegative matrices*, Linear Multilinear Algebra, **65** (2017), 1554–1565.
10. A. Melman, *Upper and lower bounds for the Perron root of a nonnegative matrix*, Linear Multilinear Algebra, **61** (2013), 171–181.
11. R. Xing, B. Zhou, *Sharp bounds for the spectral radius of nonnegative matrices*, Linear Algebra its Appl., **449** (2014), 194–209.
12. C. Wen, T. Z. Huang, *A modified algorithm for the Perron root of a nonnegative matrix*, Appl. Math. Comput., **217** (2011), 4453–4458.
13. P. Walters, *An introduction to ergodic theory*, New York (NY), Springer-Verlag, 1982.