Modified Riccati Equation Emanating from Lainiotis Filter

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***Abstract-* The modified Riccati equation arises in the implementation of Kalman filter in target tracking under measurement uncertainty and it cannot be transformed in an equation of the form of the Riccati equation. The modified Riccati equation emanating from Lainiotis filter is derived. It is shown that the two equations have the same behavior: if the system is stable, then there exists a steady state solution, while if the system is unstable, then there exists a critical value of the measurement detection probability, below which the modified Riccati equations diverge. It is established that this critical value increases in a logarithmic way, as the system becomes more unstable.**

Keywords- Modified Riccati Equation; Kalman Filter; Lainiotis Filter

I. INTRODUCTION

The discrete time modified Riccati equation emanating from Kalman filter was originally formulated in [6]. It plays an important role in target tracking [4]-[7], [10]-[11], [13]. Theoretical properties of the modified Riccati equation have been derived in [2] and [4]. It is well known [4] that the modified Riccati equation cannot be transformed in an equation of the form of the Riccati equation. The discrete time Riccati equation arises in linear estimation, namely in the implementation of the discrete time Kalman filter [1], [8]. The modified Riccati equation is solvable under certain conditions [4], [11] and has existence and uniqueness properties similar to the Riccati equation [3]. In this paper, the modified Riccati equation associated with target tracking under measurement uncertainty is studied; the case without clutter, but with detection probability of less than one is considered. The modified Riccati equation emanating from Lainiotis filter [9] is also derived and compared to the modified Riccati equation emanating from Kalman filter; it is shown that the two equations have the same behavior.

II. SYSTEM MODEL

Consider the following state space equations for $k \geq 0$:

$$x_{k+1} = Fx_k + w_k \tag{1}$$

$$z_k = Hx_k + v_k \tag{2}$$

where x_k is the $n \times 1$ dimensional state vector at time k, z_k is the $m \times 1$ dimensional measurement vector, F is the $n \times n$ system transition matrix, H is the $m \times n$ output matrix, $\{w_k\}$ and $\{v_k\}$ are independent Gaussian zero-mean white and uncorrelated random processes, Q is the $n \times n$ plant noise covariance matrix, R is the plant measurement noise covariance matrix. At the initial time $k = 0$, the state x_0 is independent of the processes $\{w_k\}$ and $\{v_k\}$ and x_0 is a Gaussian random process with mean $\bar{x}_0$ and covariance P_0.

These equations can be used in target tracking to describe a linear target motion and measurement model. In addition, consider that a measurement is received with detection probability p_d [4], where

$$0 \leq p_d \leq 1 \tag{3}$$

III. THE MODIFIED RICCATI EQUATION EMANATING FROM KALMAN FILTER

Using the Kalman filter equations [1], [8] we are able to derive [4], [13] a Kalman-like recursion for the symmetric $n \times n$ dimentional prediction error covariance matrix $P_{k+1/k}$, the **modified Riccati equation emanating from Kalman filter** (**modREKF**):

$$\begin{aligned} P_{k+1/k} &= Q + FP_{k/k-1}F^T \\ &\quad - p_d FP_{k/k-1}H^T[HP_{k/k-1}H^T + R]^{-1}HP_{k/k-1}F^T \end{aligned} \tag{4}$$

with initial condition $P_{0/-1} = P_0$. It is known [3] that the solution of the modified Riccati equation emanating from Kalman filter is independent of the initial condition $P_{0/-1}$. So, for convenience, it is reasonable to use $P_0 = 0$. Then, the initial condition for the prediction error covariance matrix becomes $P_{0/-1} = 0$.

Note that the non-singularity of $HP_{k/k-1}H^T + R$ in modREKF is guaranteed if R is a positive definite ($R > 0$) matrix.

- Setting $p_d = 1$ in (4) the (classical) *Riccati equation* emanating from Kalman filter (REKF) is derived:

$$\begin{aligned} P_{k+1/k} &= Q + FP_{k/k-1}F^T \\ &\quad - FP_{k/k-1}H^T[HP_{k/k-1}H^T + R]^{-1}HP_{k/k-1}F^T \end{aligned} \tag{5}$$

The difference between modREKF and REKF is the term of detection probability p_d. It is obvious that modREKF (4) cannot be transformed in REKF (5).

- Setting $p_d = 0$ in (4) the (classical) *Lyapunov equation* emanating from Kalman filter (LEKF) is derived:

$$P_{k+1/k} = Q + FP_{k/k-1}F^T \tag{6}$$

Note that LEKF is derived from REKF in the infinite measurement noise case ($R \to \infty$).

Concerning the modified Riccati equation emanating from Kalman filter, it is known [11] that:

- for stable systems (all eigenvalues of F lie inside the unit circle), the modified Riccati equation emanating from Kalman filter always converges and the limiting value P_p of the prediction error covariance is the steady state solution of the discrete time modified Riccati equation emanating from Kalman filter
- for unstable systems (there is at least one eigenvalue of F that lies strictly outside the unit circle), there exists a critical value of detection probability p_d , below which the modified Riccati equation emanating from Kalman filter diverges.

IV. THE MODIFIED RICCATI EQUATION EMANATING FROM LAINIOTIS FILTER

Using the Lainiotis filter equations [9] we are able to derive a Lainiotis-like recursion for the symmetric $n \times n$ dimentional estimation error covariance matrix $P_{k/k}$, the **modified Riccati equation emanating from Lainiotis filter** (**modRELF**):

$$P_{k+1/k+1} = P_n + p_d F_n [I + P_{k/k} O_n]^{-1} P_{k/k} F_n^T + (1 - p_d) F P_{k/k} F^T \qquad (7)$$

with initial condition $P_{0/0} = P_0 - p_d P_0 H^T [H P_0 H^T + R]^{-1} H P_0$,

where the following constant matrices are calculated off-line:

$$P_n = Q - p_d Q H^T A H Q \qquad (8)$$

$$F_n = F - Q H^T A H F \qquad (9)$$

$$O_n = F^T H^T A H F \qquad (10)$$

$$A = [H Q H^T + R]^{-1} \qquad (11)$$

The solution of the modified Riccati equation emanating from Lainiotis filter is independent of the initial condition $P_{0/0}$. So, for convenience, it is reasonable to use $P_0 = 0$. Then the initial condition for the prediction error covariance matrix becomes $P_{0/0} = P_0 = 0$.

Note that the non-singularity of $I + P_{k/k} O_n$ in modRELF is guaranteed due to the precence of the identity matrix I . Also, the non-singularity of A in (11) is guaranteed if R is positive definite ($R > 0$).

- Setting $p_d = 1$ in (7) the (classical) *Riccati equation* emanating from Lainiotis filter (RELF) is derived:

$$P_{k+1/k+1} = P_n + F_n [I + P_{k/k} O_n]^{-1} P_{k/k} F_n^T \qquad (12)$$

The difference between modRELF and RELF is the term of detection probability p_d . It is obvious that modREKF (7) cannot be transformed in REKF (12).

- Setting $p_d = 0$ in (7) the (classical) *Lyapunov equation* emanating from Lainiotis filter (LELF) is derived:

$$P_{k+1/k+1} = P_n + F P_{k/k} F^T \qquad (13)$$

Note that LELF is derived from RELF in the infinite measurement noise case ($R \to \infty$), since then $A = 0$, $P_n = Q$, $F_n = F$, $O_n = 0$.

The estimation error covariance equation emanating from Lainiots filter and the prediction error covariance equation emanating from Kalman filter are related by the following equations:

$$P_{k+1/k} = Q + F P_{k/k} F^T \qquad (14)$$

and

$$P_{k+1/k+1} = P_{k+1/k} - p_d F P_{k+1/k} H^T [H P_{k+1/k} H^T + R]^{-1} H P_{k+1/k} \qquad (15)$$

In the infinite measurement noise case ($R \to \infty$), we have:

$$P_{k+1/k+1} = P_{k+1/k} \qquad (16)$$

The limiting value P_e of the estimation error covariance is the steady state solution of the discrete time modified Riccati equation emanating from Lainiots filter. The relations between the steady state solution of modREKF and modREKF are:

$$P_p = Q + F P_e F^T \qquad (17)$$

$$P_e = P_p - p_d P_p H^T [H P_p H^T + R]^{-1} H P_p \qquad (18)$$

In the infinite measurement noise case ($R \to \infty$), we have:

$$P_p = P_e \qquad (19)$$

which is the steady state solution of the Luapunov equation.

The modified Riccati Equation (7) emanating from Lainiotis filter (modRELF) is equivalent to the modified Riccati Equation (4) emanating from Kalman filter (modREKF), in the sense that they calculate theoretically the prediction error covariance $P_{k+1/k}$ and the estimation error covariance $P_{k/k}$ that satisfy (14). In fact, from (7) and using (8)-(11) we are able to derive (4) after some algebra. Then, of course, the steady state prediction error covariance P_p and the steady state estimation error covariance P_e satisfy (17).

V. COMPUTATIONALCOMPARISON OF ALGORITHMS

The algorithms for solving the modified Riccati equation emanating from Kalman/Lainiotis filter consist of the direct implementation of the corresponding recursion modREKF/modRELF. So, all algorithms are recursive ones. Thus, the total computational time required for the implementation of each algorithm is:

$$t_a = B_a \cdot S_a \cdot t_o \qquad (20)$$

where B_a is the per recursion calculation burden required for the on-line calculations of each algorithm, S_a is the number of recursions (steps) that each algorithm executes and t_o is the time required to perform a scalar operation.

Note that modREKF and modRELF are equivalent to each other with respect to their behavior: they calculate theoretically the prediction error covariance $P_{k+1/k}$ and the estimation error covariance $P_{k/k}$ that satisfy (14) and the steady state prediction error covariance P_p and the steady state estimation error covariance P_e that satisfy (17). Then, it is reasonable to assume that both algorithms compute the limiting solution of the modified Riccati equation (or the corresponding modified Lyapunov equation) executing the same number of recursions, depending on the desired accuracy. Thus, in order to compare the algorithms with respect to their computational time, we have to compare their per recursion calculation burden required for the on-line calculations; the calculation burden of the off-line calculations (intialization process) is not taken into account.

The computational analysis is based on the analysis in [3]: scalar operations are involved in matrix manipulation operations, which are needed for the implementation of the filtering algorithms. Table I summarizes the calculation burden of needed matrix operations.

TABLE I CALCULATION BURDEN OF MATRIX OPERATIONS

Matrix Operation	Calcualation Burden
$A(n\times n)+B(n\times n)=C(n\times n)$	n^2
$A(n\times n)+B(n\times n)=S(n\times n)$ $S:symmetric$	$\frac{1}{2}(n^2+n)$
$I(n\times n)+A(n\times n)=B(n\times n)$ $I:identity$	n
$a\cdot S(n\times n)$ $a:\ constant$ $S:symmetric$	$\frac{1}{2}(n^2+n)$
$A(n\times m)\cdot B(m\times k)=C(n\times k)$	$2nmk-nk$
$A(n\times m)\cdot B(m\times n)=S(n\times n)$ $S:symmetric$	$n^2m+nm-\frac{1}{2}(n^2+n)$
$[A(n\times n)]^{-1}=B(n\times n)$	$\frac{1}{6}(16n^3-3n^2-n)$

The (per recursion) computational requirements of the recursive algorithms for solving the modified Riccati equations emanating from Kalman and Lainiotis filters, i.e. of modREKF and modRELF are computed in Tables II and III, respectively.

TABLE II MODIFIED RICCATI EQUATION – KALMAN FILTER (modREKF)

Matrix Operation	Matrix Dimensions	Calcualation Burden
$HP_{k/k-1}$	$(m\times n)\cdot(n\times n)$	$2n^2m-nm$
$HP_{k/k-1}H^T$	$(m\times n)\cdot(n\times m)$	$nm^2+nm-\frac{1}{2}(m^2+m)$
$HP_{k/k-1}H^T+R$	$(m\times m)+(m\times m)$	$\frac{1}{2}(m^2+m)$
$[HP_{k/k-1}H^T+R]^{-1}$	$m\times m$	$\frac{1}{6}(16m^3-3m^2-m)$
$[HP_{k/k-1}H^T+R]^{-1}HP_{k/k-1}$	$(m\times m)\cdot(m\times n)$	$2nm^2-nm$
$W=P_{k/k-1}H^T[HP_{k/k-1}H^T+R]^{-1}HP_{k/k-1}$	$(n\times m)\cdot(m\times n)$	$n^2m+nm-\frac{1}{2}(n^2+n)$
$p_d\cdot W$	$n\times n$	$\frac{1}{2}(n^2+n)$
$P_{k/k-1}-p_d\cdot W$	$(n\times n)+(n\times n)$	$\frac{1}{2}(n^2+n)$
$F(P_{k/k-1}-p_d\cdot W)$	$(n\times n)\cdot(n\times n)$	$2n^3-n^2$
$F(P_{k/k-1}-p_d\cdot W)F^T$	$(n\times n)\cdot(n\times n)$	$n^3+\frac{1}{2}(n^2-n)$
$P_{k/k-1}=Q+F(P_{k/k-1}-p_d\cdot W)F^T$	$(n\times n)+(n\times n)$	$\frac{1}{2}(n^2+n)$
Total	$B_{\text{mod}REKF}=\frac{1}{2}(6n^3+n^2+n)+3n^2m+3nm^2+\frac{1}{6}(16m^3-3m^2-m)$	

TABLE III MODIFIED RICCATI EQUATION – LAINIOTIS FILTER (modRELF)

Matrix Operation	Matrix Dimensions	Calcualation Burden
$P_{k/k}O_n$	$(n\times n)\cdot(n\times n)$	$2n^3-n^2$
$[I+P_{k/k}O_n]$	$(n\times n)+(n\times n)$	n
$[I+P_{k/k}O_n]^{-1}$	$n\times n$	$\frac{1}{6}(16n^3-3n^2-n)$
$[I+P_{k/k}O_n]^{-1}P_{k/k}$	$(n\times n)\cdot(n\times n)$	$n^3+\frac{1}{2}(n^2-n)$
$F_n[I+P_{k/k}O_n]^{-1}P_{k/k}$	$(n\times n)\cdot(n\times n)$	$2n^3-n^2$
$W=F_n[I+P_{k/k}O_n]^{-1}P_{k/k}F_n^T$	$(n\times n)\cdot(n\times n)$	$n^3+\frac{1}{2}(n^2-n)$
$p_d\cdot W$	$n\times n$	$\frac{1}{2}(n^2+n)$
$FP_{k/k}$	$(n\times n)\cdot(n\times n)$	$2n^3-n^2$
$V=FP_{k/k}F^T$	$(n\times n)\cdot(n\times n)$	$n^3+\frac{1}{2}(n^2-n)$
$(1-p_d)\cdot V$	$n\times n$	$\frac{1}{2}(n^2+n)$
$W+V$	$(n\times n)+(n\times n)$	$\frac{1}{2}(n^2+n)$
$P_{k+1\mid k+1}=P_n+W+V$	$(n\times n)+(n\times n)$	$\frac{1}{2}(n^2+n)$
Total		$B_{\text{mod}RELF}=\frac{1}{3}(35n^3+4n)$

The computational requirements of the recursive algorithms for solving the modified Riccati equations emanating from Kalman and Lainiotis filters are summarized in Table IV.

TABLE IV PER RECURSION CALCULATION BURDEN OF ALGORITHMS

Algorithm	Calcualation Burden
modREKF	$B_{\text{mod}REKF}=\frac{1}{2}(6n^3+n^2+n)+3n^2m+3nm^2+\frac{1}{6}(16m^3-3m^2-m)$
modRELF	$B_{\text{mod}RELF}=\frac{1}{3}(35n^3+4n)$

From Table IV, we derive the following conclusions:

1. The per recursion calculation burden of modREKF depends on the state dimension n and on the measurement dimension m, while the per recursion calculation burden of modRELF depends only on the state dimension n.

2. Concerning the modified Riccati equation solution algorithms, modRELF may be faster than modREKF; in fact modRELF is faster than modREKF if the following relation holds:

$$B_{\text{mod}REKF}-B_{\text{mod}RELF}$$
$$=\frac{1}{6}(-52n^3+3n^2-5n)+(3n^2m+3nm^2)+\frac{1}{6}(16m^3-3m^2-m)$$
$$=\frac{8}{3}\left(m^3+\frac{18n-3}{16}m^2+\frac{18n^2-1}{16}m-\frac{52n^3-3n^2+5n}{16}\right)>0$$

Consider the cubic function of m:

$$\hat{B}(m)=m^3+\frac{18n-3}{16}m^2+\frac{18n^2-1}{16}m-\frac{52n^3-3n^2+5n}{16}$$
$$=m^3+a_2m^2+a_1m+a_0$$

Then by [12, p. 362-365] the solution of the cubic equation $\hat{B}(m)=0$ is related to the intermediate variables

$$\hat{q}=\frac{1}{9}(3a_1-a_2^2)=\frac{1}{768}(180n^2+36n-19)$$

$$\hat{r} = \tfrac{1}{54}(9a_1a_2 - 27a_0 - 2a_2^3)$$
$$= \tfrac{1}{110592}(197208n^3 - 11340n^2 + 15498n + 243)$$

and the discriminant $\Delta = \hat{q}^3 + \hat{r}^2$.

Since $\hat{q} > 0$, it is obvious that $\Delta > 0$; thus there exist two complex conjugate roots $\rho_1(n)$, $\rho_2(n)$ and one real root $\rho_r(n)$ [12, p. 364]. Moreover, the real root $\rho_r(n)$ is a positive number, since $\rho_1(n)\rho_2(n) = \rho_1(n)\overline{\rho_1(n)} = |\rho_1(n)|^2$ and

$$\rho_1(n)\rho_2(n)\rho_r(n) = \frac{52n^3 - 3n^2 + 5n}{16} > 0 \text{ for every } n \geq 1.$$

Hence, if $m > \rho_r(n)$, it is implied that $\hat{B}(m) > 0$. Thus, when $m > \rho_r(n)$, the faster filter is modRELF, while when $1 \leq m \leq \rho_r(n)$, the faster filter is modREKF.

Fig. 1 depicts the relation between the dimensions n and m that may hold in order to decide which algorithm is faster.

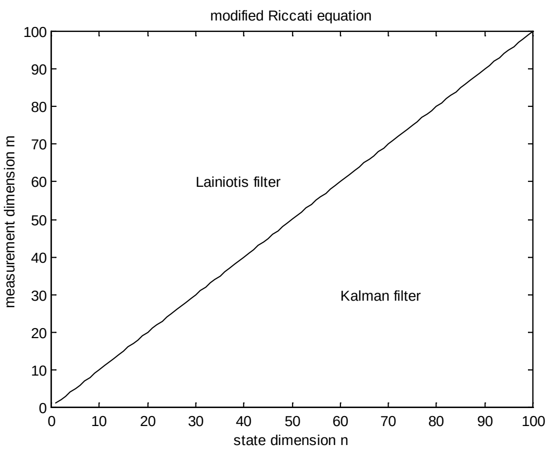

Fig. 1 ModRELF may be faster than modREKF

Then, we are able to establish the following Rule of Thumb: modRELF may be faster than modREKF; in fact modRELF is faster than modREKF if the following relation holds:

$$m > n \tag{21}$$

VI. CONVERGENCE OF THE MODIFIED RICCATI EQUATION

Simulations results were taken concerning the modified Riccati equation emanating from Kalman/Lainiotis filter. Both modREKF and modRELF were implemented in order to solve the modified Riccati equation emanating from Kalman/Lainiotis filter. Various stable as well as unstable models were considered; the maximum absolute eigenvalue of F was assumed to belong in the range [0, 10]. The following results were confirmed:

- The modified Riccati equation emanating from Lainiotis filter has the same behavior as the modified Riccati equation emanating from Kalman filter.
- If the system is stable (all eigenvalues of F lie inside the unit circle), then the modified Riccati equation emanating from Kalman/Lainiotis filter always converges: there always exists a steady state solution.
- If the system is unstable (there is at least one eigenvalue of F that lies strictly outside the unit circle), then there exists a critical value p_d^c of detection probability p_d, below which the modified Riccati equation emanating from Kalman/Lainiotis filter diverges.
- The critical value p_d^c of detection probability p_d increases in a logarithmic way as the maximum absolute eigenvalue of F increases, i.e. the system becomes more unstable. Fig. 2 depicts the relation between the system stability and the modified Riccati equation convergence.

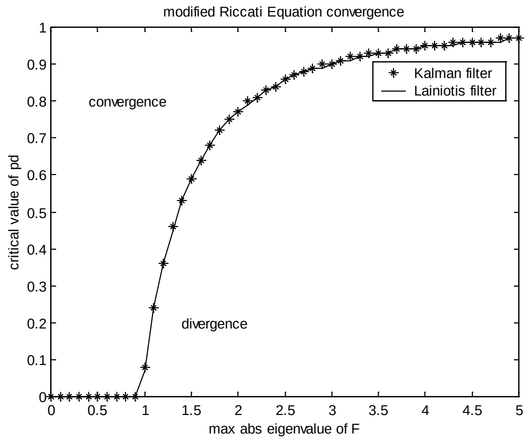

Fig. 2 Modified Riccati equation convergence w.r.t. system stability

- The modified Riccati equation emanating from Lainiotis filter behaves a slightly better than the modified Riccati equation emanating from Kalman filter, since $p_d^c(LF) \leq p_d^c(KF)$ in few cases. One of these cases happens when the system is unstable and the maximum absolute eigenvalue of F lies on the unit circle; then the critical value p_d^c takes its minimum value $p_{d,\min}^c(KF) = 0.08$ and $p_{d,\min}^c(LF) = 0.07$.
- The maximum absolute eigenvalue of F, $\lambda_{\max}(F)$ below, which the modified Riccati equation always diverges, is of the order of $|\lambda_{\max}(F)| = 10$ and it does not depend on the value of p_d; in fact $|\lambda_{\max}(F)| = 9.7$.

VII. CONCLUSIONS

In this paper, the modified Riccati equation associated with target tracking under measurement uncertainty was studied; the case without clutter, but with detection probability of less than one was considered. The modified Riccati equation emanating from Lainiotis filter was derived and compared to the modified Riccati equation emanating from Kalman filter. The properties and the computational requirements of the both equations were presented in detail.

The two equations were compared in terms of convergence and requirements. The two equations have similar convergence behavior but they have different computational requirements. In fact, if the system is stable, then there exists a steady state solution, while if the system is unstable, then there exists a critical value of the measurement detection probability, below which the modified Riccati equations diverge; it is established that this critical value increases in a logarithmic way, as the system becomes more unstable. It is also established that the modified Riccati equation emanating from Lainiotis filter is faster than the modified Riccati equation emanating from Kalman filter for systems where the measurement dimension is greater than the state dimension. This is a critical issue in a multi-sensor environment for detection and tracking, especially when information processing must be performed inside a small wireless sensor mote with limited resources, providing the basis for further development of tracking applications in this area.

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