



Golden section, Fibonacci sequence and the time invariant Kalman and Lainiotis filters



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ABSTRACT

We consider the discrete time Kalman and Lainiotis filters for multidimensional stochastic dynamic systems and investigate the relation between the golden section, the Fibonacci sequence and the parameters of the filters. Necessary and sufficient conditions for the existence of this relation are obtained through the associated Riccati equations. A conditional relation between the golden section and the steady state Kalman and Lainiotis filters is derived. A Finite Impulse Response (FIR) implementation of the steady state filters is proposed, where the coefficients of the steady state filter are related to the golden section. Finally, the relation between the Fibonacci numbers and the discrete time Lainiotis filter for multidimensional models is investigated.

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1. Introduction

Although the connection between the golden section with nature, arts and architecture is known for centuries, there is presently a huge interest of modern sciences in these classical theories. Particularly, researchers in computer science (measurement theory, graph theory and communication systems [20–22]) and cryptography [18] exhibit a substantial interest in these classical theories and use them in order to model phenomena in their field. The above are only a few applications of the golden section that imply a new mathematical direction which is the creation of a fascinating and beautiful subject of the “Mathematics of Harmony” [9,19] and the references therein. The relation between the discrete time Kalman filter/Lainiotis filter and the golden section is described for scalar systems in [6,8,10,11] and for special multidimensional systems in [10,12]; in [10] two cases are examined (i) the noise covariances, transition and measurement matrices are equal to the identity matrix, (ii) the output matrix is the identity matrix, and in [12] the elements of the steady state covariance and gain matrices are functions of the golden ratio.

This paper examines the situation in which the parameters of the Kalman filter [2,15] and Lainiotis filter [16] for more general multidimensional stochastic dynamic systems related to the golden section and the Fibonacci sequence, extending the results in [6,10]. Concerning the novelty of the paper, we mention that: (i) we generalize the results obtained in [10] formulating specific assumptions on the transition matrix and describing the conditional relation between the golden section and Kalman and Lainiotis filters for more general multidimensional systems and (ii) we investigate the theoretical

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properties of the transition matrix and its relation with the covariance matrices in order to be able to be guaranteed a relation between the golden section and Kalman and Lainiotis filters.

The paper is organized as follows: In Section 2, a review of the discrete time Kalman and Lainiotis filters and a necessary review of the matrix analysis are presented. In Section 3, the multidimensional stochastic dynamic system is considered. Necessary and sufficient conditions for the existence of the relation between the discrete time Kalman filter/Lainiotis filter and the golden section are obtained through the associated Riccati equations. Also a location of eigenvalues and the spectral radius of the transition matrix of the system is presented. In Section 4, the relation between the steady state Kalman filter/Lainiotis filter and the golden section is described and a Finite Impulse Response (FIR) implementation of the steady state filters is proposed, where the coefficients of the filter are related to the golden section. In Section 5, the relation between the Fibonacci numbers and the discrete time Lainiotis filter is presented. Finally, Section 6 is devoted to conclusions.

2. Notation and preliminaries

2.1. Golden section, Golden ratio and Fibonacci sequence

The terms “golden section” and “golden ratio” and as a concept has a long history in mathematics, see e.g. [6] and the references therein. *Golden section*, α , is called the positive root of the equation $\lambda^2 + \lambda - 1 = 0$, which is equal to

$$\alpha = \frac{-1 + \sqrt{5}}{2} \approx 0.618 \quad (1)$$

and *golden ratio*, ϕ , is the reciprocal of the golden section, which is given:

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618 \quad (2)$$

The relations between the golden section and the golden ratio are derived by (1), (2) and are given below:

$$\phi = \frac{1}{\alpha} = 1 + \alpha \quad (3)$$

For each $v \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ from (3) arises

$$\frac{1}{1 + \phi^v} = \frac{\alpha^v}{1 + \alpha^v}. \quad (4)$$

The Fibonacci sequence is defined by recurrence by taking each subsequent number as the sum of the two previous ones

$$f_{v+2} = f_{v+1} + f_v, \quad v \in \mathbb{Z}_+ = \{1, 2, \dots\}, \quad \text{with } f_1 = 0, \quad f_2 = 1. \quad (5)$$

Thus, the sequence of Fibonacci numbers is $\{0, 1, 1, 2, 3, 5, 8, \dots\}$, [1,17]. It is well-known that the Fibonacci sequence satisfies the limit properties [1,8]

$$\lim_{v \rightarrow \infty} \frac{f_v}{f_{v+1}} = \alpha \quad \text{and} \quad \lim_{v \rightarrow \infty} \frac{f_{v+2}}{f_{v+1}} = \phi, \quad (6)$$

which provide the relation between the Fibonacci sequence on the one hand and the golden section and the golden ratio on the other hand.

2.2. Stochastic dynamic system

Consider the time invariant stochastic dynamic system described by the following state space equations

$$\begin{aligned} x_{k+1} &= Fx_k + w_k \\ z_k &= Hx_k + v_k \end{aligned} \quad (7)$$

for $k = 0, 1, \dots$, where x_k is $n \times 1$ state vector at time k , z_k is $m \times 1$ measurement vector, F is $n \times n$ transition matrix, H is $m \times n$ output matrix, $\{w_k\}$, $\{v_k\}$ are the plant noise and the measurement noise process, respectively. These processes are assumed to be Gaussian, zero-mean, white and uncorrelated random processes with Q , R be $n \times n$ plant noise and $m \times m$ measurement noise covariance matrices, respectively. The vector x_0 is a Gaussian random process with mean \bar{x}_0 and covariance P_0 . In the sequel, consider that Q , R , P_0 are positive definite matrices. Also, x_0 , $\{w_k\}$ and $\{v_k\}$ are independent.

The filtering problem is to produce an estimate at time L of the state vector using measurements till time L , i.e., the aim is to use the measurements set $\{z_1, z_2, \dots, z_L\}$ in order to calculate an estimate value $x_{L/L}$ of the state vector x_L .

2.3. Review of the Kalman filter and Lainiotis filter

The discrete time Kalman filter [2,15] and the discrete time Lainiotis filter [16] are well-known algorithms that solve the filtering problem, by computing the estimate value $x_{k/k}$ of the state vector at time k , and the corresponding estimation error covariance matrix $P_{k/k}$. The two filters are equivalent to each other [4] since they compute theoretically the same estimations: Kalman filter computes the estimation through prediction, while Lainiotis filter computes the estimation through smoothing.

The time invariant filters are given by the following equations:

Kalman Filter

$$x_{k+1/k} = Fx_{k/k} \tag{8}$$

$$P_{k+1/k} = FP_{k/k}F^T + Q \tag{9}$$

$$K_{k+1} = P_{k+1/k}H^T[HP_{k+1/k}H^T + R]^{-1} \tag{10}$$

$$x_{k+1/k+1} = x_{k+1/k} + K_{k+1}z_{k+1} - K_{k+1}Hx_{k+1/k} \tag{11}$$

$$P_{k+1/k+1} = P_{k+1/k} - K_{k+1}HP_{k+1/k} \tag{12}$$

for $k = 0, 1, \dots$, with initial conditions $P_{0/0} = P_0$, and $x_{0/0} = x_0$.

Lainiotis Filter

$$P_{k+1/k+1} = P_n + F_n[I + P_{k/k}O_n]^{-1}P_{k/k}F_n^T \tag{13}$$

$$x_{k+1/k+1} = F_n[I + P_{k/k}O_n]^{-1}x_{k/k} + K_nz_{k+1} + F_n[I + P_{k/k}O_n]^{-1}P_{k/k}K_mz_{k+1} \tag{14}$$

for $k = 0, 1, \dots$, with initial conditions $P_{0/0} = P_0$, $x_{0/0} = x_0$, and the following constant matrices are calculated off-line:

$$A = [HQH^T + R]^{-1} \tag{15}$$

$$K_n = QA^T \tag{16}$$

$$K_m = F^T A^T \tag{17}$$

$$P_n = Q - K_nHQ \tag{18}$$

$$F_n = F - K_nHF \tag{19}$$

$$O_n = K_mHF \tag{20}$$

It is well known [2] that if the signal process system is asymptotically stable (i.e., all the eigenvalues of F lie inside the unit circle), there exists a unique positive definite steady state value P_p of the prediction error covariance matrix, i.e., the prediction error covariance $P_{k+1/k}$ converges to the steady state prediction error covariance

$$P_p = \lim_{k \rightarrow \infty} P_{k+1/k}. \tag{21}$$

The Riccati equation emanating from Kalman filter is derived by the Eqs. (10), (12) in (9) and given by

$$P_{k+2/k+1} = Q + FP_{k+1/k}F^T - FP_{k+1/k}H^T[HP_{k+1/k}H^T + R]^{-1}HP_{k+1/k}F^T \tag{22}$$

and due to (21) the steady state prediction error covariance matrix is a positive definite solution of (22), i.e., P_p satisfies

$$P_p = Q + FP_pF^T - FP_pH^T[HP_pH^T + R]^{-1}HP_pF^T. \tag{23}$$

Moreover, there exists a unique positive definite steady state value P_e of the estimation error covariance matrix, i.e. the estimation error covariance $P_{k/k}$ converges to the steady state estimation error covariance

$$P_e = \lim_{k \rightarrow \infty} P_{k/k}. \tag{24}$$

The Riccati equation emanating from Lainiotis filter is given by (13) and due to (24) the steady state estimation error covariance matrix is the positive definite solution, which satisfies the following equation

$$P_e = P_n + F_n[I + P_eO_n]^{-1}P_eF_n^T, \tag{25}$$

where P_n , F_n , O_n are given by (18)–(20). Also, the steady state estimation error covariance matrix, P_e , arises from (12) of Kalman filter, when P_p and the steady state Kalman filter gain

$$K = \lim_{k \rightarrow \infty} K_{k+1} \tag{26}$$

are given.

Furthermore, there exists a unique positive definite steady state value P_s of the smoothing error covariance matrix, i.e., the smoothing error covariance $P_{k/k+1}$ converges to the steady state smoothing error covariance

$$P_s = \lim_{k \rightarrow \infty} P_{k/k+1}. \quad (27)$$

The relation between the smoothing error covariance matrix and the estimation error covariance matrix can be calculated from fixed-point and fixed-lag smoothing algorithms [2] emanating from Kalman filter equation

$$P_{k/k+1} = P_{k/k} + P_{k/k} F^T P_{k+1/k}^{-1} (P_{k+1/k+1} - P_{k+1/k}) P_{k+1/k}^{-1} F P_{k/k}. \quad (28)$$

Using (21), (24) and (27) arises the limiting solution of (28), which is the steady state smoothing error covariance matrix P_s , and satisfies the following equation

$$P_s = P_e + P_e F^T P_p^{-1} (P_e - P_p) P_p^{-1} F P_e. \quad (29)$$

Since the two filters are equivalent, the relation between the smoothing error covariance matrix and the estimation error covariance matrix is obtained by the Lainiotis filter equation

$$P_{k/k+1} = [I + P_{k/k} O_n]^{-1} P_{k/k} = [P_{k/k}^{-1} + O_n]^{-1}, \quad (30)$$

which leads to

$$P_s = [I + P_e O_n]^{-1} P_e = (P_e^{-1} + O_n)^{-1}, \quad (31)$$

where O_n is given by (20).

Since the equivalence of the two filters is confirmed in [4], in the following one of the two time invariant implementation is used and it is referred as Kalman filter/Lainiotis filter.

2.4. Review of the matrix analysis

For an $n \times n$ real symmetric matrix A the symbol $A > 0$ ($A \geq 0$) means that the matrix A is positive definite (semidefinite). The notation $A > B$ ($A \geq B$) means that A and B are $n \times n$ real symmetric matrices and $A - B > 0$ ($A - B \geq 0$).

To describe our results, we need the following lemmas.

Lemma 2.1 [2, Matrix Inversion Lemma]. *Let A , C be nonsingular matrices and let B be $n \times m$ matrix. Then, the following equation holds*

$$[A + BCB^T]^{-1} = A^{-1} - A^{-1}B[C^{-1} + B^T A^{-1}B]^{-1}B^T A^{-1}.$$

Lemma 2.2 [14, Observation 7.1.6]. *Let A be $n \times n$ symmetric positive definite matrix and let B be a real $n \times m$ matrix. Then $B^T A B$ is positive semidefinite matrix, with $\text{rank}(B^T A B) = \text{rank}(B)$. Moreover, $B^T A B$ is positive definite matrix if and only if $\text{rank}(B) = m$.*

Lemma 2.3 [14, Theorem 7.2.6]. *Let A be $n \times n$ symmetric positive (semi) definite matrix and let $\mu \in \mathbb{Z}_+$ be a given positive integer. Then there exists a unique symmetric positive (semi) definite matrix B such that $B^\mu = A$. The unique positive (semi) definite root of A is denoted by $B = \sqrt[\mu]{A}$ for each $\mu \in \mathbb{Z}_+$, and the following relations hold*

$$AB = BA \text{ and } B^T = (\sqrt[\mu]{A})^T = \sqrt[\mu]{A^T} = \sqrt[\mu]{A} = B.$$

The most useful case of this Lemma 2.3 is for $\mu = 2$. The unique positive (semi) definite square root of the positive (semi) definite matrix A is denoted by $B = \sqrt{A}$.

Lemma 2.4. *Let A , B be $n \times n$ real symmetric matrices.*

- (i) [14, Corollary 4.5.18 and Theorem 4.1.5] *There exists a real orthogonal matrix U such that the matrices A , B are simultaneously diagonalizable, that is $UAU^T = D_A$ and $UBU^T = D_B$, if and only if $AB = BA$, if and only if AB is symmetric.*
- (ii) *Let A , $B > 0$ with $AB = BA$. Then, $\sqrt{A}\sqrt{B} = \sqrt{AB}$.*

Let A be $n \times n$ matrix. The set of all $\lambda \in \mathbb{C}$ that are eigenvalues of A is called the *spectrum* of A and denoted by $\sigma(A)$. The *spectral radius* of A is the nonnegative real number $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

Lemma 2.5. *Let A , B be $n \times n$ real symmetric matrices with $A > 0$.*

- (i) [14, Theorem 7.6.3] *The product AB is a diagonalizable matrix, all of whose eigenvalues are real. The matrix AB has the same number of positive, negative and zero eigenvalues as B .*
- (ii) [14, Theorem 7.7.3] *Suppose $B \geq 0$. Then $A \geq B$ if and only if $\rho(BA^{-1}) \leq 1$ and $A > B$ if and only if $\rho(BA^{-1}) < 1$, where $\rho(BA^{-1})$ is the spectral radius of BA^{-1} .*

Lemma 2.6. Let A be $n \times n$ positive definite matrix and B be $n \times n$ positive semidefinite matrix. Then, the matrix equation

$$XAX^T = B \quad (32)$$

has

$$X = \sqrt{B}\sqrt{A^{-1}} \quad (33)$$

as a solution.

Proof. Since $A > 0$, $B \geq 0$, there exist the unique square roots of the associated matrices A , B . Also, the matrix X in (33) satisfies (32) since using the properties by Lemma 2.3 arises

$$XAX^T = \sqrt{B}\sqrt{A^{-1}}\sqrt{A}\sqrt{A}(\sqrt{B}\sqrt{A^{-1}})^T = \sqrt{B}\sqrt{A^{-1}}\sqrt{A}\sqrt{A}(\sqrt{A^{-1}})^T(\sqrt{B})^T = \sqrt{B}\sqrt{B} = B. \quad \square$$

3. The covariance matrices and the golden section

In this section assume the time invariant stochastic dynamic system described in (7), where the output matrix H be $m \times n$, with $m \geq n$. Also the $n \times n$ plant noise covariance matrix Q is assumed to be $Q > 0$, and the $m \times m$ measurement noise covariance matrix R is assumed to be $R > 0$; it is evident that $R^{-1} > 0$, due to $R > 0$.

Consider the $n \times n$ matrix

$$S = H^T R^{-1} H. \quad (34)$$

According to Lemma 2.2 the strictly positive definiteness of S in (34) is equivalent to $\text{rank}(H) = n$, with $m \geq n$, hence the singularity of the matrix S depends on the rank and the size of H ; thus here and hereafter, we suppose that the size of the output matrix H is $m \times n$ with $m \geq n$. In this case it is evident that S^{-1} is well defined with $S^{-1} > 0$.

The main result of the section is based on the following Theorem 3.1, in which the relations between the steady state prediction/estimation/smoothing error covariance matrix, the steady state Kalman filter gain, the matrices Q , R , H on the one hand and the golden section α and the golden ratio ϕ on the other hand are presented.

Theorem 3.1. Let Q , S , $\alpha^v S^{-1} - Q$ be $n \times n$ positive definite matrices and $R > 0$ be $m \times m$ matrix, where $S = H^T R^{-1} H$ in (34) and the output matrix H be $m \times n$ with $m \geq n$. For any given $v \in \mathbb{Z}^* = \{\pm 1, \pm 2, \dots\}$, the Riccati equation emanating from Kalman filter in (23) has a positive definite solution,

$$P_p = \alpha^v S^{-1} \quad (35)$$

if and only if the $n \times n$ transition matrix F is formulated by

$$F = \sqrt{1 + \phi^v} \sqrt{\alpha^v S^{-1} - Q} \sqrt{S}. \quad (36)$$

Moreover, the steady state Kalman filter gain is formulated by

$$K = \frac{\alpha^v}{1 + \alpha^v} S^{-1} H^T R^{-1} = \frac{1}{1 + \phi^v} S^{-1} H^T R^{-1}, \quad (37)$$

the steady state estimation error covariance matrix is given by

$$P_e = \frac{\alpha^v}{1 + \alpha^v} S^{-1} = \frac{1}{1 + \phi^v} S^{-1}, \quad (38)$$

and the steady state smoothing error covariance matrix is given by

$$P_s = \frac{\alpha^v}{1 + \alpha^v} S^{-1} - \frac{\alpha^v}{(1 + \alpha^v)^2} \left(\sqrt{S} \sqrt{\alpha^v S^{-1} - Q} \sqrt{S^{-1}} \right)^T \sqrt{S} \sqrt{\alpha^v S^{-1} - Q} \sqrt{S^{-1}}. \quad (39)$$

Proof. Using Lemma 2.1 and setting $S = H^T R^{-1} H$ the Riccati Eq. (23) is written as

$$\begin{aligned} P_p &= Q + FP_p F^T - FP_p H^T \left\{ R^{-1} - R^{-1} H [P_p^{-1} + H^T R^{-1} H]^{-1} H^T R^{-1} \right\} H P_p F^T \\ &= Q + FP_p F^T - FP_p S P_p F^T + FP_p S [P_p^{-1} + S]^{-1} S P_p F^T. \end{aligned} \quad (40)$$

Substituting in the right part of (40) the prediction matrix P_p by (35) we have

$$\begin{aligned} P_p &= Q + \alpha^v FS^{-1} F^T - \alpha^{2v} FS^{-1} SS^{-1} F^T + \alpha^{2v} FS^{-1} S[\alpha^{-v} S + S]^{-1} SS^{-1} F^T = Q + \alpha^v FS^{-1} F^T - \alpha^{2v} FS^{-1} F^T + \frac{\alpha^{3v}}{1 + \alpha^v} FS^{-1} F^T \\ &= Q + \frac{\alpha^v}{1 + \alpha^v} FS^{-1} F^T. \end{aligned} \quad (41)$$

Considering the hypothesis that P_p in (35) solves the Riccati Eq. (23), the matrix Eq. (41) is written as

$$\alpha^v S^{-1} - Q = \frac{\alpha^v}{1 + \alpha^v} FS^{-1} F^T, \quad (42)$$

which is of type (32). Since $\alpha^v S^{-1} - Q > 0$, and $\frac{\alpha^v}{1 + \alpha^v} S^{-1} > 0$, Lemma 2.6 is applicable, and so the solution of (42) formulates as

$$F = \sqrt{\alpha^v S^{-1} - Q} \sqrt{\left(\frac{\alpha^v}{1 + \alpha^v}\right)^{-1}} S,$$

which yields F in (36) due to (4).

Conversely, substituting in the matrix Eq. (41) the matrix F by (36) we have

$$P_p = Q + \frac{\alpha^v}{1 + \alpha^v} (1 + \phi^v) \sqrt{\alpha^v S^{-1} - Q} \sqrt{SS^{-1}} \left(\sqrt{\alpha^v S^{-1} - Q} \sqrt{S} \right)^T = Q + \sqrt{\alpha^v S^{-1} - Q} \sqrt{SS^{-1}} \sqrt{S} \sqrt{\alpha^v S^{-1} - Q} \quad (43)$$

$$= Q + \sqrt{\alpha^v S^{-1} - Q} \sqrt{S} \sqrt{S^{-1}} \sqrt{S^{-1}} \sqrt{S} \sqrt{\alpha^v S^{-1} - Q} \quad (44)$$

$$= Q + \sqrt{\alpha^v S^{-1} - Q} \sqrt{\alpha^v S^{-1} - Q} = \alpha^v S^{-1}.$$

Hence, P_p in (35) satisfies the Riccati equation in (23) under the choice of the transition matrix F by (36). Note that the equality in (43) arises, since $\alpha^v S^{-1} - Q > 0$, and according to Lemma 2.3 holds $\left(\sqrt{\alpha^v S^{-1} - Q}\right)^T = \sqrt{\alpha^v S^{-1} - Q}$. Also, the equality in (44) is derived, since $S > 0$ it follows that $S^{-1} > 0$; hence there exists a unique symmetric positive definite square root of S^{-1} such that $S^{-1} = \sqrt{S^{-1}} \sqrt{S^{-1}}$, (see, Lemma 2.3).

Since the Kalman filter gain converges to the steady state Kalman filter gain by (26), combining the limiting Eq. (10) and (21), using the matrix inversion lemma (see, Lemma 2.1) and (35), the steady state Kalman filter gain is derived as follows

$$\begin{aligned} K &= \lim_{k \rightarrow \infty} K_{k+1} = \lim_{k \rightarrow \infty} P_{k+1/k} H^T [H \lim_{k \rightarrow \infty} P_{k+1/k} H^T + R]^{-1} = P_p H^T [HP_p H^T + R]^{-1} = P_p H^T R^{-1} - P_p S [P_p^{-1} + S]^{-1} H^T R^{-1} \\ &= \alpha^v S^{-1} H^T R^{-1} - \alpha^v S^{-1} S [\alpha^{-v} S + S]^{-1} H^T R^{-1} = \frac{\alpha^v}{1 + \alpha^v} S^{-1} H^T R^{-1}. \end{aligned} \quad (45)$$

The proof for the steady state Kalman filter gain in (37) is completed using in (45) the relation of the golden section/ratio by (4).

According to (24) the steady state estimation error covariance matrix can be derived by (12), where the matrices P_p , K are formulated by (35) and (37), as follows

$$\begin{aligned} P_e &= \lim_{k \rightarrow \infty} P_{k+1/k+1} = \lim_{k \rightarrow \infty} P_{k+1/k} - \lim_{k \rightarrow \infty} K_{k+1} H \lim_{k \rightarrow \infty} P_{k+1/k} = P_p - KHP_p = \alpha^v S^{-1} - \frac{\alpha^{2v}}{1 + \alpha^v} S^{-1} H^T R^{-1} HS^{-1} \\ &= \alpha^v S^{-1} - \frac{\alpha^{2v}}{1 + \alpha^v} S^{-1} = \frac{\alpha^v}{1 + \alpha^v} S^{-1}. \end{aligned}$$

The steady state smoothing error covariance matrix can be derived by (29), where the matrices P_e , P_p , F are formulated by (38), (35) and (36), respectively, as follows

$$\begin{aligned} P_s &= P_e + P_e F^T P_p^{-1} (P_e - P_p) P_p^{-1} F P_e = \frac{1}{1 + \phi^v} S^{-1} + \frac{1}{(1 + \phi^v)^2} S^{-1} F^T (\alpha^v S^{-1})^{-1} \left(\frac{1}{1 + \phi^v} S^{-1} - \alpha^v S^{-1} \right) (\alpha^v S^{-1})^{-1} F S^{-1} \\ &= \frac{1}{1 + \phi^v} S^{-1} - \frac{1}{(1 + \alpha^v)(1 + \phi^v)^2} S^{-1} F^T S F S^{-1} \\ &= \frac{1}{1 + \phi^v} S^{-1} - \frac{1}{(1 + \alpha^v)(1 + \phi^v)} \left(\sqrt{S^{-1}} \right)^2 \left(\sqrt{\alpha^v S^{-1} - Q} \sqrt{S} \right)^T S \sqrt{\alpha^v S^{-1} - Q} \sqrt{S} \left(\sqrt{S^{-1}} \right)^2 \\ &= \frac{1}{1 + \phi^v} S^{-1} - \frac{1}{(1 + \alpha^v)(1 + \phi^v)} \sqrt{S^{-1}} \sqrt{\alpha^v S^{-1} - Q} \sqrt{S} \sqrt{S} \sqrt{\alpha^v S^{-1} - Q} \sqrt{S^{-1}} \end{aligned} \quad (46)$$

which completes the proof of Theorem 3.1. \square

Remark 3.1

- (i) The equivalence of the two filters allows us to use the Riccati equation in (25) in order to prove the formula of the steady state estimation error covariance matrix P_e in (38). Also, the formula of the steady state smoothing error covariance matrix P_s in (39) can be proved using (31) emanating from Lainiotis filter.
- (ii) For the time invariant stochastic dynamic system in (7) consider the scalar case $m = n = 1$. It has been proved in [6, Section 7] that supposing an inequality for the coefficients F, H, Q, R (see (41) in [6]) as the assumptions of Theorem 3.1, the steady state estimation error covariance matrix P_e of Lainiotis filter is related to the golden section, if and only if the transition coefficient F is given by the golden section and the coefficients H, Q, R (see (43) in [6]) as this formulated for the associated matrices in Theorem 3.1 with $v = -1$. Also, the formulas F, P_e coincide with the associated matrices in (36) and (38), when $v = -1$. Thus, Theorem 3.1 is a generalization of the scalar case in [6, Section 7].

Corollary 3.2. Let the output matrix H be $n \times n$, and let $Q, S, R, \alpha^v S^{-1} - Q$ be $n \times n$ positive definite matrices with S, F in (34), (36), respectively. For any given $v \in \mathbb{Z}^*$ the steady state Kalman filter gain is formulated by

$$K = \frac{\alpha^v}{1 + \alpha^v} H^{-1} = \frac{1}{1 + \phi^v} H^{-1}. \tag{47}$$

Proof. According to Lemma 2.2 the assumption $S > 0$ yields $\text{rank}(H) = n$, which is equivalent to $\det(H) \neq 0$. Thus the non-singularity of H allows us to write

$$S^{-1} = [H^T R^{-1} H]^{-1} = H^{-1} R H^{-T}.$$

The validity of (47) follows from the above equality and (37) of Theorem 3.1. \square

Remark 3.2. Suppose $m = n$ for the time invariant stochastic dynamic system in (7) with $H = I$ and $Q = R > 0$. From (34) arises $S = R^{-1}$, whereby it is evident that $\alpha^v S^{-1} - Q = (\alpha^v - 1)R > 0$ if and only if $v < 0$. In this case using (36) the transition matrix is given by

$$F = \sqrt{1 + \phi^v} \sqrt{(\alpha^v - 1)R} \sqrt{R^{-1}} = \sqrt{\frac{\alpha^{2v} - 1}{\alpha^v}} I.$$

- (i) Consider $v = -1$. Then, the above equality implies $F = I$, and using (3) the equalities (35), (47), (or (37)–(39)) yield

$$P_p = \phi R, \quad K = \frac{\alpha^{-1}}{1 + \alpha^{-1}} I = \alpha I, \tag{48}$$

$$P_e = \frac{\alpha^{-1}}{1 + \alpha^{-1}} R = \alpha R, \quad P_s = \frac{2\alpha}{(1 + \alpha)^2} R = 2\alpha^3 R. \tag{49}$$

Notice that, the matrices P_p and K in (48) generalize the results in [10, Lemmas 2, 3], where a multidimensional stochastic dynamic system as in (7) with standard form considered, i.e., $Q = R = F = H = I$.

- (ii) Consider the scalar case $m = n = 1$, with $v = -1$, which is known as *random walk system*. Then, the steady state prediction/estimation/smoothing error covariance matrix in (48), (49) coincides with the associated coefficient in [6, Table 2].

In the following propositions the eigenvalues of the transition matrix F are located and some conditions are established, which are required such that F be a symmetric matrix.

Proposition 3.3. Let Q, S, R , and $\alpha^v S^{-1} - Q$ be positive definite matrices as in Theorem 3.1 and the transition matrix F in (36). For any given $v \in \mathbb{Z}^*$, the matrix F has real positive eigenvalues with

$$\rho(F) < 1 + \alpha^v, \tag{50}$$

i.e., the spectral radius of F is related to the golden section α .

Proof. Since the matrices $\alpha^v S^{-1} - Q > 0$ and $S > 0$, it follows that the square roots of the matrices are positive definite (see, Lemma 2.3), hence F is written as product two positive definite matrices as its square roots. According to Lemma 2.5 (i) F is a diagonalizable matrix with its eigenvalues be real positive numbers as the eigenvalues of the square roots.

Moreover, since $Q, S^{-1} > 0$, the following relations are derived

$$\alpha^{2\nu}S^{-1} + Q > 0 \Rightarrow \alpha^{2\nu}S^{-1} + Q + \alpha^\nu S^{-1} - \alpha^\nu S^{-1} > 0 \Rightarrow \alpha^{2\nu}S^{-1} + \alpha^\nu S^{-1} > \alpha^\nu S^{-1} - Q \Rightarrow \alpha^\nu(1 + \alpha^\nu)S^{-1} > \alpha^\nu S^{-1} - Q > 0$$

and according to [Lemma 2.3](#) follows

$$\sqrt{\alpha^\nu(1 + \alpha^\nu)S^{-1}} > \sqrt{\alpha^\nu S^{-1} - Q} > 0.$$

Thus, [Lemma 2.5](#) (ii) is applicable, and so $\rho(\sqrt{\alpha^\nu S^{-1} - Q}[\sqrt{\alpha^\nu(1 + \alpha^\nu)S^{-1}}]^{-1}) < 1$, which implies

$$\rho\left(\frac{1}{\sqrt{\alpha^\nu(1 + \alpha^\nu)}}\sqrt{\alpha^\nu S^{-1} - Q}\sqrt{S}\right) < 1.$$

Also, since $\frac{1}{1+\alpha^\nu}F = \frac{1}{1+\alpha^\nu}\sqrt{1 + \phi^\nu}\sqrt{\alpha^\nu S^{-1} - Q}\sqrt{S} = \frac{1}{\sqrt{\alpha^\nu(1+\alpha^\nu)}}\sqrt{\alpha^\nu S^{-1} - Q}\sqrt{S}$, the preceding inequality can be written

$$\rho\left(\frac{1}{1 + \alpha^\nu}F\right) < 1,$$

which leads to [\(50\)](#) due to $\alpha^\nu > 0$. \square

Proposition 3.4. Let Q, S, R and $\alpha^\nu S^{-1} - Q$ be positive definite matrices, and $QS = SQ$. Then, for any given $\nu \in \mathbb{Z}^*$, the transition matrix F in [Theorem 3.1](#) is formulated

$$F = \sqrt{1 + \phi^\nu}\sqrt{\alpha^\nu I - QS}. \quad (51)$$

Moreover, F is a positive definite matrix.

Proof. The assumption of the commutativity of the matrices S, Q yields

$$(\alpha^\nu S^{-1} - Q)S = S(\alpha^\nu S^{-1} - Q),$$

that means $\alpha^\nu S^{-1} - Q, S$ are commutative matrices; applying [Lemma 2.4](#) (ii) we obtain

$$\sqrt{\alpha^\nu S^{-1} - Q}\sqrt{S} = \sqrt{S(\alpha^\nu S^{-1} - Q)} = \sqrt{\alpha^\nu I - QS}. \quad (52)$$

The validity of [\(51\)](#) follows from [\(52\)](#) and [\(36\)](#).

The commutativity of the matrices and [\(52\)](#) yield

$$\begin{aligned} F^T &= \sqrt{1 + \phi^\nu}\left(\sqrt{\alpha^\nu S^{-1} - Q}\sqrt{S}\right)^T = \sqrt{1 + \phi^\nu}\sqrt{S}\sqrt{\alpha^\nu S^{-1} - Q} = \sqrt{1 + \phi^\nu}\sqrt{S(\alpha^\nu S^{-1} - Q)} = \sqrt{1 + \phi^\nu}\sqrt{\alpha^\nu I - QS} \\ &= \sqrt{1 + \phi^\nu}\sqrt{\alpha^\nu I - QS} = F. \end{aligned}$$

Hence, F is a symmetric matrix and according to [Proposition 3.3](#) its eigenvalues are real positive numbers. Thus $F > 0$. \square

Remark 3.3

- (i) According to [Lemma 2.4](#) (i) the commutativity of Q, S in [Proposition 3.4](#) is necessary in order to $\alpha^\nu I - QS$ is a symmetric matrix. Also, the assumptions of definiteness of the matrices guarantees $\alpha^\nu I - QS > 0$, since $\alpha^\nu I - QS = (\alpha^\nu S^{-1} - Q)S$; consequently the square root of $\alpha^\nu I - QS$ is well defined and unique, (see, [Lemma 2.3](#)).
- (ii) Since F in [\(51\)](#) depends on the commutativity of Q, S , using the assumptions of [Proposition 3.4](#) and working as in the proof of [Theorem 3.1](#), the same formulas of P_p, K, P_e in [\(35\), \(37\)](#) and [\(38\)](#) can be proved. The commutativity of Q, S effects the formula of P_s ; in particular, the substitution of F by [\(51\)](#) in [\(46\)](#) yields

$$P_s = \frac{\alpha^\nu}{(1 + \alpha^\nu)^2}(S^{-1} + Q). \quad (53)$$

- (iii) Consider $\nu = 1$, and $H = I$; it is obvious that $\text{rank}(H) = n$, and $S = R^{-1} > 0$ by [\(34\)](#). According to [Proposition 3.4](#), if $\alpha R - Q > 0$ and $RQ = QR$, then, using [\(3\)](#), the matrix F in [\(51\)](#) is written as

$$F = \sqrt{1 + \phi}\sqrt{\alpha I - QR^{-1}} = \sqrt{\alpha + 1}\sqrt{I - \frac{1}{\alpha}QR^{-1}} = \sqrt{\phi}\sqrt{I - \frac{1}{\alpha}QR^{-1}}.$$

Using the associated formulas from [\(35\), \(37\)](#) (or [\(47\)](#) due to $\det(H) \neq 0$), [\(38\)](#) and [\(53\)](#) we obtain

$$P_p = \alpha R, \quad K = \frac{\alpha}{1 + \alpha}I = \alpha^2 I,$$

$$P_e = \frac{\alpha}{1 + \alpha}R = \alpha^2 R, \quad P_s = \frac{\alpha}{(1 + \alpha)^2}(R + Q) = \alpha^3(R + Q).$$

The matrices P_p and K are given in [10, Proposition 6] using the preceding matrix F , without the necessary restriction of the commutativity of the matrices Q, R , which guarantees $\alpha I - QR^{-1} > 0$, (see, Proposition 3.4 and Lemma 2.4).

(iv) Consider $v = -1$, and $H = I$, then $S = R^{-1}$. According to Proposition 3.4, if $\phi R - Q > 0$ and $RQ = QR$, then F in (51) is written as

$$F = \sqrt{1 + \phi^{-1}} \sqrt{\alpha^{-1}I - QR^{-1}} = \sqrt{1 + \alpha} \sqrt{\frac{1}{\alpha}(I - \alpha QR^{-1})} = \sqrt{\phi + 1} \sqrt{I - \frac{1}{\phi} QR^{-1}}.$$

Using the associated formulas from (35), (37), (38) and (53) we obtain

$$P_p = \phi R, \quad K = \frac{\alpha^{-1}}{1 + \alpha^{-1}} I = \alpha I,$$

$$P_e = \frac{\alpha^{-1}}{1 + \alpha^{-1}} R = \alpha R, \quad P_s = \frac{\alpha^{-1}}{(1 + \alpha^{-1})^2} (R + Q) = \alpha^3 (R + Q).$$

The matrices P_p and K are given in [10, Proposition 7] using the preceding matrix F , without the necessary restriction of the commutativity of the matrices Q, R , which guarantees $\phi I - QR^{-1} > 0$, (see, Proposition 3.4).

Corollary 3.5. Let Q, S, R and $\alpha^v S^{-1} - Q$ be positive definite matrices as in Theorem 3.1 and let F be the transition matrix in (36). For any given $v \in \mathbb{Z}^*$, holds

$$P_p > P_e > P_s, \tag{54}$$

where P_p, P_e, P_s are formulated by (35), (38), (39), respectively.

Proof. By the formulas (35), (38) and $S^{-1} > 0$, it is evident that holds

$$P_p - P_e = \alpha^v S^{-1} - \frac{\alpha^v}{1 + \alpha^v} S^{-1} = \left(\alpha^v - \frac{\alpha^v}{1 + \alpha^v} \right) S^{-1} = \frac{\alpha^{2v}}{1 + \alpha^v} S^{-1} > 0,$$

it is implied $P_p > P_e$.

Also, using (38) and (46) we have

$$P_e - P_s = \frac{\alpha^v}{(1 + \phi^v)(1 + \alpha^v)^2} S^{-1} F^T S F S^{-1}. \tag{55}$$

According to Proposition 3.3, F has real positive eigenvalues, it is implied $\det(F) \neq 0 \iff \text{rank}(F) = n$ and $\det(FS^{-1}) \neq 0 \iff \text{rank}(FS^{-1}) = n$. Hence, according to Lemma 2.2 for $S > 0$ and $\text{rank}(FS^{-1}) = n$, it follows $S^{-1} F^T S F S^{-1} = (FS^{-1})^T S F S^{-1} > 0$. Thus, (55) yields $P_e > P_s$, which completes the proof. \square

4. The steady state filters and the golden section

In this section the relations between the steady state $x_{k+1/k+1}$ on the one hand and the golden section α and the golden ratio ϕ on the other hand are presented, under the suitable choice of the transition matrix F .

Theorem 4.1. Let $Q, S, R, \alpha^v S^{-1} - Q$ be positive definite matrices as in Theorem 3.1, let H be $m \times n$ with $m \geq n$ and F in (36). For any given $v \in \mathbb{Z}^*$, the recursive form of the steady state Kalman filter/Lainiotis filter is formulated by

$$x_{k+1/k+1} = \frac{1}{1 + \alpha^v} F x_{k/k} + \frac{1}{1 + \phi^v} S^{-1} H^T R^{-1} z_{k+1}, \tag{56}$$

for $k = 0, 1, \dots$, with initial condition $x_{0/0} = x_0$.

Proof. Working as in [5, Appendix A] and using the Eqs. (8)–(12) of Kalman filter arises

$$x_{k+1/k+1} = P_{k+1/k+1} [F P_{k/k} F^T + Q]^{-1} F x_{k/k} + P_{k+1/k+1} H^T R^{-1} z_{k+1}, \tag{57}$$

for $k = 0, 1, \dots$, with initial condition $x_{0/0} = x_0$. In the steady state case, since the estimation error covariance matrices converge to P_e , in the matrix Eq. (57) substituting $P_{k+1/k+1}, P_{k/k}$ with P_e by (38), using (41) and setting P_p as in (35), the recursive form of the steady state Kalman filter/Lainiotis filter is derived

$$\begin{aligned} x_{k+1/k+1} &= P_e [F P_e F^T + Q]^{-1} F x_{k/k} + P_e H^T R^{-1} z_{k+1} = \frac{\alpha^v}{1 + \alpha^v} S^{-1} \left[\frac{\alpha^v}{1 + \alpha^v} F S^{-1} F^T + Q \right]^{-1} F x_{k/k} + \frac{\alpha^v}{1 + \alpha^v} S^{-1} H^T R^{-1} z_{k+1} \\ &= \frac{\alpha^v}{1 + \alpha^v} S^{-1} P_p^{-1} F x_{k/k} + \frac{\alpha^v}{1 + \alpha^v} S^{-1} H^T R^{-1} z_{k+1} = \frac{1}{1 + \alpha^v} F x_{k/k} + \frac{\alpha^v}{1 + \alpha^v} S^{-1} H^T R^{-1} z_{k+1}. \end{aligned}$$

The validity of (56) follows from the above equality and (4). \square

Remark 4.1. Notice that the equivalence of the two filters allows us to use (57) in order to prove the recursive form of the steady state Kalman filter/Lainiotis filter in (56); the same formula may be proved using only either the Eq. (14) of the Lainiotis filter or (11) and (8) of the Kalman filter.

Consider $m = n$ in Theorem 4.1 and using (56), we can prove as in Corollary 3.2 the following result for the recursive form of the steady state Kalman filter/Lainiotis filter, since H is nonsingular matrix.

Corollary 4.2. Let the output matrix H be $n \times n$, and let Q , S , R , $\alpha^v S^{-1} - Q$ be $n \times n$ positive definite matrices with S , F in (34), (36), respectively. For any given $v \in \mathbb{Z}^*$ the recursive form of the steady state Kalman filter/Lainiotis filter is formulated by

$$x_{k+1/k+1} = \frac{1}{1 + \alpha^v} F x_{k/k} + \frac{1}{1 + \alpha^v} H^{-1} z_{k+1}, \quad (58)$$

for $k = 0, 1, \dots$, with initial condition $x_{0/0} = x_0$.

Taking advantage of the recursive form of the steady state Kalman filter/Lainiotis filter $x_{k+1/k+1}$ by (56) of Theorem 4.1, the closed form of the steady state Kalman filter/Lainiotis filter is presented in the following proposition.

Theorem 4.3. Let Q , S , R and $\alpha^v S^{-1} - Q$ be positive definite matrices as in Theorem 3.1 and F in (36). For any given $v \in \mathbb{Z}^*$, the closed form of the steady state Kalman filter/Lainiotis filter is formulated

$$x_{k+1/k+1} = \frac{1}{(1 + \alpha^v)^{k+1}} F^{k+1} x_0 + \sum_{i=1}^{k+1} \frac{\alpha^v}{(1 + \alpha^v)^{k+2-i}} F^{k+1-i} S^{-1} H^T R^{-1} z_i, \quad (59)$$

for $k = 0, 1, \dots$, and $x_{0/0} = x_0$.

Proof. The proof is based on the induction method. In fact, for $k = 0$ in (56) the recursive form of the steady state Kalman filter/Lainiotis filter is written as

$$x_{1/1} = \frac{1}{1 + \alpha^v} F x_0 + \frac{1}{1 + \alpha^v} S^{-1} H^T R^{-1} z_1 = \frac{1}{1 + \alpha^v} F x_0 + \frac{\alpha^v}{1 + \alpha^v} S^{-1} H^T R^{-1} z_1,$$

which satisfies (59). Assume that (59) is true for k ; then, using the assumption of induction and (56) it is derived

$$\begin{aligned} x_{k+2/k+2} &= \frac{1}{1 + \alpha^v} F x_{k+1/k+1} + \frac{\alpha^v}{1 + \alpha^v} S^{-1} H^T R^{-1} z_{k+2} \\ &= \frac{1}{1 + \alpha^v} F \left\{ \frac{1}{(1 + \alpha^v)^{k+1}} F^{k+1} x_0 + \sum_{i=1}^{k+1} \frac{\alpha^v}{(1 + \alpha^v)^{k+2-i}} F^{k+1-i} S^{-1} H^T R^{-1} z_i \right\} + \frac{\alpha^v}{1 + \alpha^v} S^{-1} H^T R^{-1} z_{k+2} \\ &= \frac{1}{(1 + \alpha^v)^{k+2}} F^{k+2} x_0 + \sum_{i=1}^{k+2} \frac{\alpha^v}{(1 + \alpha^v)^{k+3-i}} F^{k+2-i} S^{-1} H^T R^{-1} z_i. \end{aligned}$$

i.e., (59) holds also for $k + 1$, and hence for all $k \geq 0$. \square

Remark 4.2. By (59) of Theorem 4.3 it is evident that the closed form of the steady state Kalman filter/Lainiotis filter computes the state estimate at time $k + 1$ as a linear combination of the initial state estimate and of all the previous measurements with coefficients related to the golden section and golden ratio, in fact it is required all the $k + 1$ previous measurements.

In the following proposition, generalizing the results of Theorem 4.3, a generic closed form of the steady state Kalman filter/Lainiotis filter after N steps is given.

Proposition 4.4. Let Q , S , R and $\alpha^v S^{-1} - Q$ be positive definite matrices as in Theorem 3.1 and F in (36). For any given $v \in \mathbb{Z}^*$, and a given nonnegative integer k , the recursive form of the steady state Kalman filter/Lainiotis filter after N steps is formulated

$$x_{k+N/k+N} = \left(\frac{1}{1 + \alpha^v} F \right)^N x_{k/k} + \frac{1}{1 + \alpha^v} \sum_{i=1}^N \left(\frac{1}{1 + \alpha^v} F \right)^{N-i} S^{-1} H^T R^{-1} z_{k+i} \quad (60)$$

and the associated closed form of the steady state Kalman filter/Lainiotis filter is formulated

$$x_{k+N/k+N} = \left(\frac{1}{1+\alpha^v}F\right)^{k+N} x_0 + \frac{1}{1+\phi^v} \sum_{i=1}^{k+N} \left(\frac{1}{1+\alpha^v}F\right)^{k+N-i} S^{-1}H^T R^{-1}z_i, \tag{61}$$

for $N \in \mathbb{Z}_+$ and $x_{0/0} = x_0$.

Proof. The Eq. (60) can be proved by induction. For $N = 1$ in (60) arises (56). Suppose that (60) is true for N ; then, using the assumption of induction and (56) we derive

$$\begin{aligned} x_{k+N+1/k+N+1} &= x_{(k+1)+N/(k+1)+N} = \left(\frac{1}{1+\alpha^v}F\right)^N x_{k+1/k+1} + \frac{1}{1+\phi^v} \sum_{i=1}^N \left(\frac{1}{1+\alpha^v}F\right)^{N-i} S^{-1}H^T R^{-1}z_{k+1+i} \\ &= \left(\frac{1}{1+\alpha^v}F\right)^{N+1} x_{k/k} + \frac{1}{1+\phi^v} \left(\frac{1}{1+\alpha^v}F\right)^N S^{-1}H^T R^{-1}z_{k+1} + \frac{1}{1+\phi^v} \sum_{j=2}^{N+1} \left(\frac{1}{1+\alpha^v}F\right)^{N-j+1} S^{-1}H^T R^{-1}z_{k+j} \\ &= \left(\frac{1}{1+\alpha^v}F\right)^{N+1} x_{k/k} + \frac{1}{1+\phi^v} \sum_{i=1}^{N+1} \left(\frac{1}{1+\alpha^v}F\right)^{N+1-i} S^{-1}H^T R^{-1}z_{k+i}, \end{aligned}$$

i.e., (60) holds also for $N + 1$, and hence for all $N \geq 1$.

In (60), substituting $x_{k/k}$ by (59), using (4) and setting $\tau = k + N$, we obtain

$$\begin{aligned} x_{k+N/k+N} &= \left(\frac{1}{1+\alpha^v}F\right)^N \left\{ \left(\frac{1}{1+\alpha^v}F\right)^k x_0 + \sum_{i=1}^k \frac{\alpha^v}{(1+\alpha^v)^{k+1-i}} F^{k-i} S^{-1}H^T R^{-1}z_i \right\} \\ &\quad + \frac{1}{1+\phi^v} \sum_{i=1}^N \left(\frac{1}{1+\alpha^v}F\right)^{N-i} S^{-1}H^T R^{-1}z_{k+i} \\ &= \left(\frac{1}{1+\alpha^v}F\right)^{k+N} x_0 + \frac{1}{1+\phi^v} \sum_{i=1}^k \left(\frac{1}{1+\alpha^v}F\right)^{k+N-i} S^{-1}H^T R^{-1}z_i + \frac{1}{1+\phi^v} \sum_{i=1}^N \left(\frac{1}{1+\alpha^v}F\right)^{N-i} S^{-1}H^T R^{-1}z_{k+i} \\ &= \left(\frac{1}{1+\alpha^v}F\right)^\tau x_0 + \frac{1}{1+\phi^v} \sum_{i=1}^{\tau-N} \left(\frac{1}{1+\alpha^v}F\right)^{\tau-i} S^{-1}H^T R^{-1}z_i + \frac{1}{1+\phi^v} \sum_{j=\tau-N+1}^{\tau} \left(\frac{1}{1+\alpha^v}F\right)^{\tau-j} S^{-1}H^T R^{-1}z_j \\ &= \left(\frac{1}{1+\alpha^v}F\right)^\tau x_0 + \frac{1}{1+\phi^v} \sum_{i=1}^{\tau} \left(\frac{1}{1+\alpha^v}F\right)^{\tau-i} S^{-1}H^T R^{-1}z_i, \end{aligned} \tag{62}$$

which for $\tau = k + N$ completes the proof of Proposition 4.4. \square

Remark 4.3. It is evident that the recursive form of the steady state Kalman filter/Lainiotis filter in (60) computes the state estimate at time $k + N$ as a linear combination of all the previous estimates and the measurements with coefficients related to the golden section and golden ratio, in fact it is required all the k previous estimates and the $k + N$ measurements.

Moreover, the closed form of the steady state Kalman filter/Lainiotis filter in (61) computes the state estimate at time $k + N$ as a linear combination of the initial state estimate and of all previous $k + N$ measurements with coefficients related to the golden section/ratio.

At this point, for a given integer v , combining the spectral property of matrix $\frac{1}{1+\alpha^v}F$ by Proposition 3.3 with the conclusion in [13, rule of thumb, p. 356], “if the spectral radius of $\frac{1}{1+\alpha^v}F$ is less than 1, then the computed powers of $\frac{1}{1+\alpha^v}F$ can be expected to converge to zero”, we are able to write

$$\lim_{N_0 \rightarrow \infty} \left(\frac{1}{1+\alpha^v}F\right)^{N_0} = 0,$$

which means that there exists some $N \in \mathbb{Z}_+$ such that

$$\left(\frac{1}{1+\alpha^v}F\right)^N \neq 0 \text{ and } \left(\frac{1}{1+\alpha^v}F\right)^{N+j} < E, \quad \forall j = 1, 2, \dots \tag{63}$$

for an arbitrary matrix E , whose elements are small positive numbers. Using the ideas in [3,7], basing on the formulated property in (63) and Proposition 4.4, in the following theorem we are able to derive a Finite Impulse Response (FIR) form of the steady state Kalman filter/Lainiotis filter.

Theorem 4.5. Let Q, S, R and $\alpha^v S^{-1} - Q$ be positive definite matrices as in Theorem 3.1 and F in (36). For any given $v \in \mathbb{Z}^*$, the FIR form of the steady state Kalman filter/Lainiotis filter is formulated

$$x_{\tau/\tau} = \frac{1}{1 + \phi^v} \sum_{i=0}^N \frac{1}{(1 + \alpha^v)^{N-i}} F^{N-i} S^{-1} H^T R^{-1} z_{\tau-N+i}, \tag{64}$$

where $\tau, N \in \mathbb{Z}_+$ with $\tau > N$, and N is verified the relations in (63).

Proof. Using the closed form in (62) for $\tau > N$ by Proposition 4.4, we derive

$$\begin{aligned} x_{\tau/\tau} &= \left(\frac{1}{1 + \alpha^v} F\right)^\tau x_0 + \frac{1}{1 + \phi^v} \sum_{j=1}^\tau \left(\frac{1}{1 + \alpha^v} F\right)^{\tau-j} S^{-1} H^T R^{-1} z_j \\ &= \left(\frac{1}{1 + \alpha^v} F\right)^\tau x_0 + \frac{1}{1 + \phi^v} \left\{ \sum_{j=1}^{\tau-N-1} \left(\frac{1}{1 + \alpha^v} F\right)^{\tau-j} S^{-1} H^T R^{-1} z_j + \sum_{j=\tau-N}^\tau \left(\frac{1}{1 + \alpha^v} F\right)^{\tau-j} S^{-1} H^T R^{-1} z_j \right\} \\ &= \left(\frac{1}{1 + \alpha^v} F\right)^\tau x_0 + \frac{1}{1 + \phi^v} \sum_{j=1}^{\tau-N-1} \left(\frac{1}{1 + \alpha^v} F\right)^{\tau-j} S^{-1} H^T R^{-1} z_j + \frac{1}{1 + \phi^v} \sum_{i=0}^N \left(\frac{1}{1 + \alpha^v} F\right)^{N-i} S^{-1} H^T R^{-1} z_{\tau-N+i} \end{aligned} \tag{65}$$

The property of $\frac{1}{1 + \alpha^v} F$ in (63) and the assumption $\tau > N$ yield that in (65) the coefficient of x_0 converges to zeros due to $\tau \geq N + 1$ and all the coefficients of z_j of the sum for $1 \leq j \leq \tau - N - 1$ converge to zero, since $N + 1 \leq \tau - j \leq \tau - 1 \Rightarrow \tau - j \geq N + 1$. Thus, (64) follows from (65). \square

Notice that, using (4), the Eq. (64) yields

$$x_{\tau/\tau} = \sum_{i=0}^N \frac{\alpha^v}{(1 + \alpha^v)^{N-i+1}} F^{N-i} S^{-1} H^T R^{-1} z_{\tau-N+i},$$

whereby we conclude that the proposed FIR implementation of the steady state Kalman filter/Lainiotis filter computes the state estimate as a linear combination of a known number of the last measurements with coefficients $z_{\tau-N+i}$, which are related to the golden section.

5. Lainiotis filter and Fibonacci sequence

In this section generalizing the results concerning a scalar time invariant stochastic dynamic system in [6], we present the relation between the Fibonacci numbers and the discrete time Lainiotis filter for multidimensional filtering models, completing the study for the filters, since the associated results for the Kalman filter have been presented in [10].

Consider that the time invariant stochastic dynamic system in (7) with the transition and output matrix equal to the $n \times n$ identity, $F = H = I$, i.e.,

$$\begin{aligned} x_{k+1} &= x_k + w_k \\ z_k &= x_k + v_k \end{aligned} \tag{66}$$

and the plant noise and measurement noise process having equal the $n \times n$ noise covariances $Q = R = \Sigma^2 > 0$. Note that only this choice of the model parameters allow us to derive the relation between the Fibonacci numbers and the discrete time Lainiotis filter as in [10], where the relation between the Fibonacci numbers and the discrete time Kalman filter is derived.

In this case, the Lainiotis filter parameters by (15)–(20) are given by

$$A = K_m = O_n = [2\Sigma^2]^{-1}, \quad K_n = F_n = \frac{1}{2}I, \quad P_n = \frac{1}{2}\Sigma^2. \tag{67}$$

Proposition 5.1. *The recursive form of the Lainiotis filter is formulated*

$$P_{k+1/k+1} = \Sigma^2 [2\Sigma^2 + P_{k/k}]^{-1} (\Sigma^2 + P_{k/k}) \tag{68}$$

$$x_{k+1/k+1} = \Sigma^2 [2\Sigma^2 + P_{k/k}]^{-1} x_{k/k} + \Sigma^2 [2\Sigma^2 + P_{k/k}]^{-1} (\Sigma^2 + P_{k/k}) [\Sigma^2]^{-1} z_{k+1}, \tag{69}$$

for $k = 0, 1, \dots$, with initial conditions $P_{0/0} = P_0$, and $x_{0/0} = x_0$.

Proof. Using the value of the parameter O_n by (67) we can write

$$[I + P_{k/k} O_n]^{-1} = [2\Sigma^2 [2\Sigma^2]^{-1} + P_{k/k} [2\Sigma^2]^{-1}]^{-1} = 2\Sigma^2 [2\Sigma^2 + P_{k/k}]^{-1}. \tag{70}$$

In (13) and (14) substituting the values of parameters by (67) and using (70), the estimation error covariance is written as

$$P_{k+1/k+1} = \frac{1}{2}\Sigma^2 + \frac{1}{2}\Sigma^2 [2\Sigma^2 + P_{k/k}]^{-1} P_{k/k} = \frac{1}{2}\Sigma^2 \left(I + [2\Sigma^2 + P_{k/k}]^{-1} P_{k/k} \right) = \frac{1}{2}\Sigma^2 [2\Sigma^2 + P_{k/k}]^{-1} (2\Sigma^2 + 2P_{k/k}),$$

which yields (68).

Moreover, the estimate vector $x_{k+1|k+1}$ is written as

$$\begin{aligned} x_{k+1/k+1} &= \Sigma^2 [2\Sigma^2 + P_{k/k}]^{-1} x_{k/k} + \frac{1}{2} \left(I + 2\Sigma^2 [2\Sigma^2 + P_{k/k}]^{-1} P_{k/k} [2\Sigma^2]^{-1} \right) z_{k+1} \\ &= \Sigma^2 [2\Sigma^2 + P_{k/k}]^{-1} x_{k/k} + \Sigma^2 \left([2\Sigma^2]^{-1} + [2\Sigma^2 + P_{k/k}]^{-1} P_{k/k} [2\Sigma^2]^{-1} \right) z_{k+1} \\ &= \Sigma^2 [2\Sigma^2 + P_{k/k}]^{-1} x_{k/k} + \Sigma^2 \left(I + [2\Sigma^2 + P_{k/k}]^{-1} P_{k/k} \right) [2\Sigma^2]^{-1} z_{k+1} \\ &= \Sigma^2 [2\Sigma^2 + P_{k/k}]^{-1} x_{k/k} + \Sigma^2 [2\Sigma^2 + P_{k/k}]^{-1} (2\Sigma^2 + 2P_{k/k}) [2\Sigma^2]^{-1} z_{k+1}, \end{aligned}$$

which yields (69). □

In the following, for the time invariant stochastic dynamic system in (66) it is presented the relation between the coefficients of the closed form of the estimation/smoothing/prediction error covariance matrices of the Lainiotis filter and the Fibonacci sequence.

Theorem 5.2. *The closed form of the Lainiotis filter is formulated*

$$P_{k+1/k+1} = \Sigma^2 [f_{2k+4}\Sigma^2 + f_{2k+3}P_0]^{-1} (f_{2k+3}\Sigma^2 + f_{2k+2}P_0) \tag{71}$$

$$x_{k+1/k+1} = \Sigma^2 [f_{2k+4}\Sigma^2 + f_{2k+3}P_0]^{-1} x_0 + \Sigma^2 [f_{2k+4}\Sigma^2 + f_{2k+3}P_0]^{-1} \sum_{i=1}^{k+1} (f_{2i+1}\Sigma^2 + f_{2i}P_0) [\Sigma^2]^{-1} z_i, \tag{72}$$

for $k = 0, 1, \dots$, with initial conditions $P_{0/0} = P_0$, and $x_{0/0} = x_0$.

Proof. The formula of the estimation covariance matrix in (71) can be proved by induction. Using the definition of the Fibonacci sequence by (5) and the recursive form in (68) for $k = 0$ arises

$$P_{1/1} = \Sigma^2 [2\Sigma^2 + P_0]^{-1} (\Sigma^2 + P_0) = \Sigma^2 [f_4\Sigma^2 + f_3P_0]^{-1} (f_3\Sigma^2 + f_2P_0).$$

Suppose that (71) is true for k ; then, using the hypothesis of induction and the equalities of the Fibonacci numbers $2f_{2k+v} + f_{2k+v-1} = f_{2k+v} + f_{2k+v+1} = f_{2k+v+2}$, $v \in \mathbb{Z}_+$, we have

$$\begin{aligned} \Sigma^2 + P_{k+1/k+1} &= \Sigma^2 + \Sigma^2 [f_{2k+4}\Sigma^2 + f_{2k+3}P_0]^{-1} (f_{2k+3}\Sigma^2 + f_{2k+2}P_0) \\ &= \Sigma^2 [f_{2k+4}\Sigma^2 + f_{2k+3}P_0]^{-1} (f_{2k+4}\Sigma^2 + f_{2k+3}P_0 + f_{2k+3}\Sigma^2 + f_{2k+2}P_0) \\ &= \Sigma^2 [f_{2k+4}\Sigma^2 + f_{2k+3}P_0]^{-1} (f_{2k+5}\Sigma^2 + f_{2k+4}P_0) \end{aligned} \tag{73}$$

and

$$[2\Sigma^2 + P_{k+1/k+1}]^{-1} = [f_{2k+6}\Sigma^2 + f_{2k+5}P_0]^{-1} (f_{2k+4}\Sigma^2 + f_{2k+3}P_0) [\Sigma^2]^{-1}. \tag{74}$$

Moreover, $P_{k+2/k+2}$ is derived by (68) and using (73) and (74) arises

$$\begin{aligned} P_{k+2/k+2} &= \Sigma^2 [2\Sigma^2 + P_{k+1/k+1}]^{-1} (\Sigma^2 + P_{k+1/k+1}) \\ &= \Sigma^2 [f_{2k+6}\Sigma^2 + f_{2k+5}P_0]^{-1} (f_{2k+4}\Sigma^2 + f_{2k+3}P_0) [\Sigma^2]^{-1} \Sigma^2 [f_{2k+4}\Sigma^2 + f_{2k+3}P_0]^{-1} (f_{2k+5}\Sigma^2 + f_{2k+4}P_0) \\ &= \Sigma^2 [f_{2k+6}\Sigma^2 + f_{2k+5}P_0]^{-1} (f_{2k+5}\Sigma^2 + f_{2k+4}P_0), \end{aligned}$$

i.e., (71) holds also for $k + 1$, and hence holds for all $k = 0, 1, \dots$

In order to prove (72) we need the following recursive formula of the estimate vector

$$\begin{aligned} x_{k+1/k+1} &= \Sigma^2 [f_{2k+4}\Sigma^2 + f_{2k+3}P_0]^{-1} (f_{2k+2}\Sigma^2 + f_{2k+1}P_0) [\Sigma^2]^{-1} x_{k/k} + \Sigma^2 [f_{2k+4}\Sigma^2 + f_{2k+3}P_0]^{-1} (f_{2k+3}\Sigma^2 \\ &\quad + f_{2k+2}P_0) [\Sigma^2]^{-1} z_{k+1}, \end{aligned} \tag{75}$$

which arises directly from (69) due to (73) and (74).

The formula of the closed form of the estimate vector in (72) can be proved by induction. The recursive form in (75) for $k = 0$ yields

$$\begin{aligned} x_{1/1} &= \Sigma^2 [f_4 \Sigma^2 + f_3 P_0]^{-1} (f_2 \Sigma^2 + f_1 P_0) [\Sigma^2]^{-1} x_0 + \Sigma^2 [f_4 \Sigma^2 + f_3 P_0]^{-1} (f_3 \Sigma^2 + f_2 P_0) [\Sigma^2]^{-1} z_1 \\ &= \Sigma^2 [f_4 \Sigma^2 + f_3 P_0]^{-1} x_0 + \Sigma^2 [f_4 \Sigma^2 + f_3 P_0]^{-1} (f_3 \Sigma^2 + f_2 P_0) [\Sigma^2]^{-1} z_1, \end{aligned}$$

which satisfies (72).

Suppose that (72) is true for k ; $x_{k+2/k+2}$ is derived by (75) and substituting the hypothesis of induction arises

$$\begin{aligned} x_{k+2/k+2} &= \Sigma^2 [f_{2k+6} \Sigma^2 + f_{2k+5} P_0]^{-1} (f_{2k+4} \Sigma^2 + f_{2k+3} P_0) [\Sigma^2]^{-1} x_{k+1/k+1} + \Sigma^2 [f_{2k+6} \Sigma^2 + f_{2k+5} P_0]^{-1} (f_{2k+5} \Sigma^2 \\ &\quad + f_{2k+4} P_0) [\Sigma^2]^{-1} z_{k+2} \\ &= \Sigma^2 [f_{2k+6} \Sigma^2 + f_{2k+5} P_0]^{-1} x_0 + \Sigma^2 [f_{2k+6} \Sigma^2 + f_{2k+5} P_0]^{-1} \sum_{i=1}^{k+1} (f_{2i+1} \Sigma^2 + f_{2i} P_0) [\Sigma^2]^{-1} z_i \\ &\quad + \Sigma^2 [f_{2k+6} \Sigma^2 + f_{2k+5} P_0]^{-1} (f_{2k+5} \Sigma^2 + f_{2k+4} P_0) [\Sigma^2]^{-1} z_{k+2} \\ &= \Sigma^2 [f_{2k+6} \Sigma^2 + f_{2k+5} P_0]^{-1} x_0 + \Sigma^2 [f_{2k+6} \Sigma^2 + f_{2k+5} P_0]^{-1} \sum_{i=1}^{k+2} (f_{2i+1} \Sigma^2 + f_{2i} P_0) [\Sigma^2]^{-1} z_i, \end{aligned}$$

which completes the induction for the formula of the estimate vector in (72). \square

Proposition 5.3. For $k \in \mathbb{Z}_+$, the smoothing error covariance matrix is formulated

$$P_{k/k+1} = 2\Sigma^2 [f_{2k+4} \Sigma^2 + f_{2k+3} P_0]^{-1} (f_{2k+1} \Sigma^2 + f_{2k} P_0) \quad (76)$$

and the prediction error covariance matrix is formulated

$$P_{k+1/k} = \Sigma^2 [f_{2k+2} \Sigma^2 + f_{2k+1} P_0]^{-1} (f_{2k+1} \Sigma^2 + f_{2k} P_0) + \Sigma^2. \quad (77)$$

Proof. The formula of the smoothing covariance matrix in (76) arises from the Eqs. (30), (67) and (70) as follows

$$P_{k/k+1} = [I + P_{k/k} O_n]^{-1} P_{k/k} = \left[I + P_{k/k} [2\Sigma^2]^{-1} \right]^{-1} P_{k/k} = 2\Sigma^2 [2\Sigma^2 + P_{k/k}]^{-1} P_{k/k}.$$

On the right hand of the above equation we use (74) and substitute $P_{k/k}$ from the closed form in (71) of Theorem 5.2, and thus, the proof of the formula (76) is completed.

The equivalence of two filters allows us to use the Eq. (9) of Kalman filter for $F = I$, $Q = \Sigma^2$, and substituting the estimation error covariance matrix by (71) the following equality is derived

$$P_{k+1/k} = F P_{k/k} F^T + Q = \Sigma^2 [f_{2k+2} \Sigma^2 + f_{2k+1} P_0]^{-1} (f_{2k+1} \Sigma^2 + f_{2k} P_0) + \Sigma^2,$$

which completes the proof. \square

Remark 5.1

- (i) The recursive formulas in (68), (69) and the closed form (71), (72) in Theorem 5.2 coincide with the scalar case, which has been presented in [6].
- (ii) Rewriting (71) as

$$P_{k+1/k+1} = \frac{f_{2k+3}}{f_{2k+4}} \Sigma^2 \left[\Sigma^2 + \frac{f_{2k+3}}{f_{2k+4}} P_0 \right]^{-1} \left(\Sigma^2 + \frac{f_{2k+2}}{f_{2k+3}} P_0 \right),$$

using (24) and (6) in the above equation P_e is derived by

$$P_e = \lim_{k \rightarrow \infty} P_{k+1/k+1} = \alpha \Sigma^2 \left[\Sigma^2 + \alpha P_0 \right]^{-1} (\Sigma^2 + \alpha P_0) = \alpha \Sigma^2. \quad (78)$$

Similarly, the steady state smoothing/prediction error covariance matrix can be computed by (76) and (77) of Proposition 5.3; the associated formulas are given in (82).

- (iii) Suppose $m = n$ for the time invariant stochastic dynamic system in (7) with $H = I$ and $Q = R = \Sigma^2 > 0$. From (34) arises $S = R^{-1} = [\Sigma^2]^{-1}$, whereby it is evident that $\alpha^v S^{-1} - Q = (\alpha^v - 1)\Sigma^2 > 0$ if and only if $v < 0$. In this case the assumptions of Theorem 3.1 are verified and since $\Sigma^2, [\Sigma^2]^{-1}$ are commutative matrices, the matrices F, P_s, P_p, P_e are given by (51), (53), (35), (38), respectively,

$$F = \sqrt{1 + \phi^v} \sqrt{\alpha^v I - QS} = \sqrt{\frac{\alpha^{2v} - 1}{\alpha^v}} I, \quad (79)$$

$$P_s = \frac{\alpha^v}{(1 + \alpha^v)^2} (S^{-1} + Q) = \frac{2\alpha^v}{(1 + \alpha^v)^2} \Sigma^2, \quad (80)$$

$$P_p = \alpha^v S^{-1} = \alpha^v \Sigma^2, \quad P_e = \frac{\alpha^v}{1 + \alpha^v} S^{-1} = \frac{\alpha^v}{1 + \alpha^v} \Sigma^2. \quad (81)$$

Notice that, F in (79) coincides with F in Remark 3.2.

Consider the special case $v = -1$; then, the equality (79) follows $F = I$, which justifies the choice for the value of parameter F in (66), and using (80) and (81), P_e, P_s, P_p are equal to

$$P_e = \alpha \Sigma^2, \quad P_s = 2\alpha^3 \Sigma^2, \quad P_p = \alpha^{-1} \Sigma^2 = \phi \Sigma^2. \quad (82)$$

Notice that P_e in (78) coincides with (82).

Thus, we summarize that the estimation/smoothing/prediction error covariance matrix is related to the Fibonacci sequence (see, (71), (76), (77), respectively) and the associated steady state matrix is related to the golden section/ratio as in (82) due to (6).

6. Conclusions

In this paper, it was shown that the discrete time Kalman filter/Lainiotis filter related to the golden section/ratio only under specific assumptions on the transition matrix F of a time invariant multidimensional system, which formulates as in Theorem 3.1 or Proposition 3.4. In this case, the steady state prediction/estimation/smoothing error covariance matrices and the steady state Kalman filter gain are related to the golden section/ratio. Concerning the recursive and the non-recursive forms of the steady state Kalman filter/Lainiotis filter were proved associated results. Also, a FIR implementation of the steady state Kalman filter/Lainiotis filter was proposed, where the state estimate is given as a linear combination of a well-defined set of the last measurements with coefficients, which are powers of the golden section.

Finally, consider the special multidimensional stochastic dynamic system with the transition and the output matrix equal to the identity matrix and the noise covariance matrices be equal to each other, it was shown that the prediction/estimation/smoothing error covariance matrices of the Lainiotis filter are formulated by coefficients related to the Fibonacci numbers as well as the state estimate is given as a linear combination of the initial state estimate and of all previous measurements with coefficients related to the Fibonacci numbers.

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